Compositional Inductive Invariant Based Verification of Neural Network Controlled Systems

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What is this talk about?

A **compositional** method for verifying the

safety properties of

neural network controlled systems

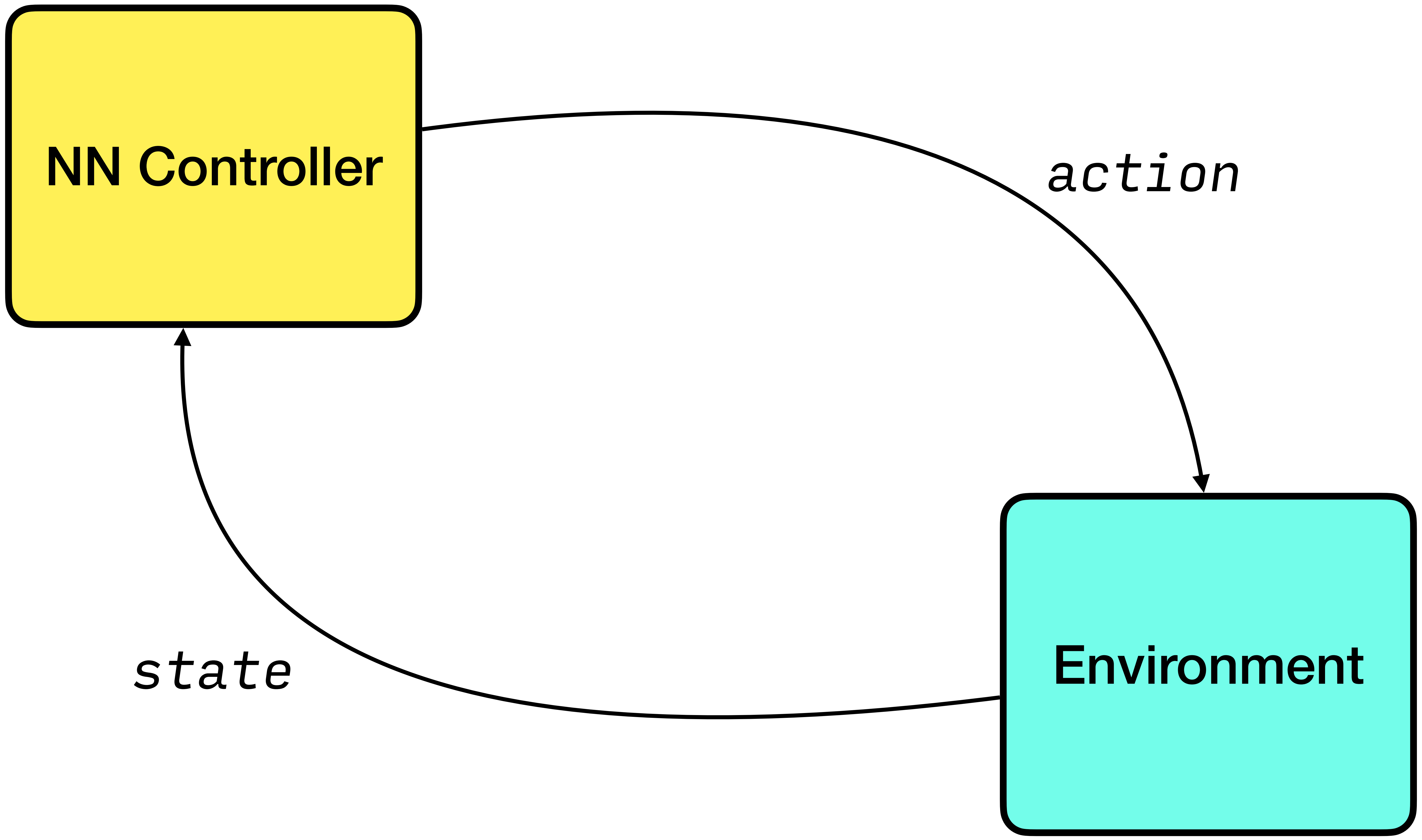
over an **infinite time horizon**

using **inductive invariant method**

Neural Network Controlled Systems

(NNCS)

Neural Network Controlled Systems



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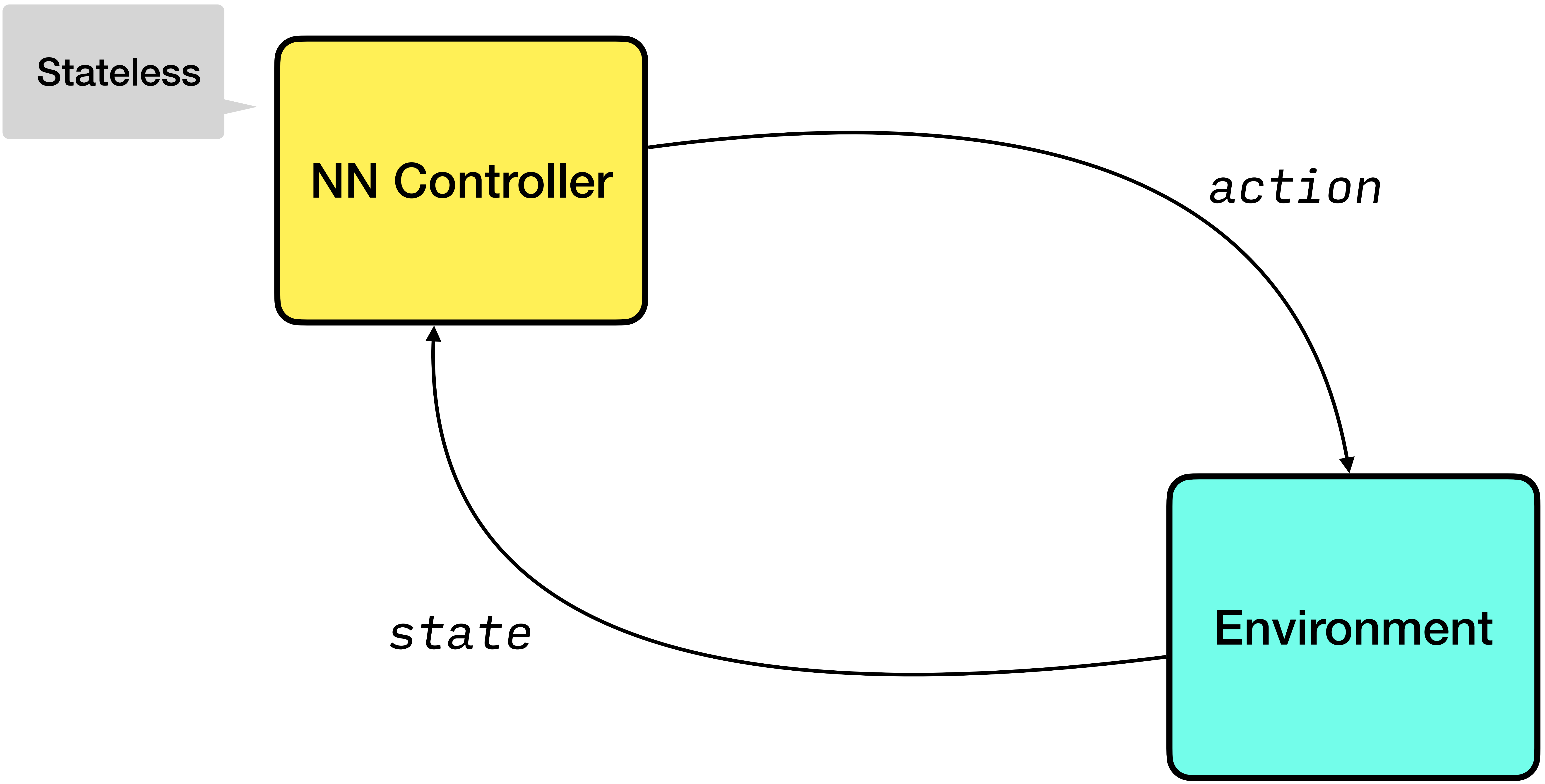

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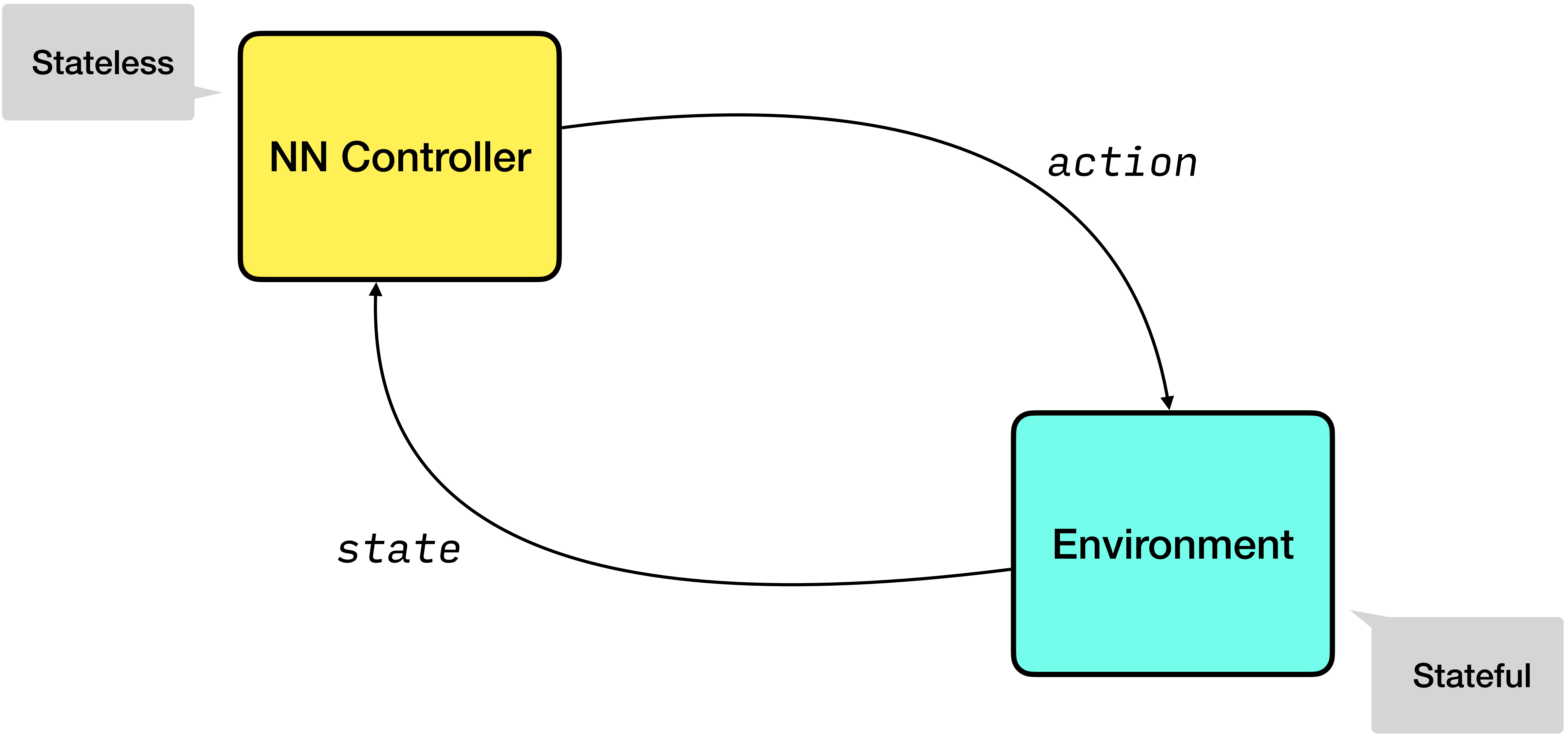
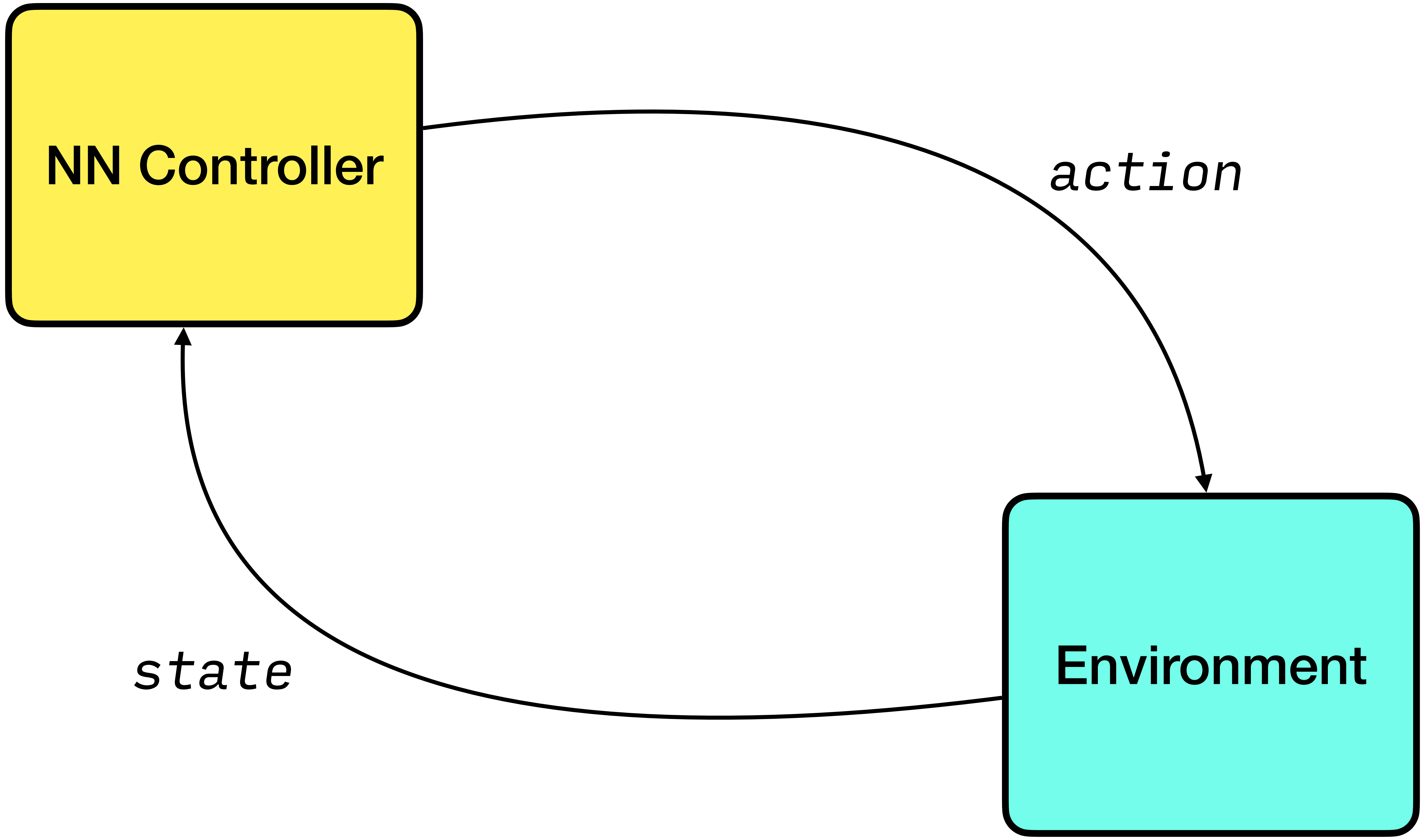


NNCS Safety Verification Problem



Check that every reachable `state` satisfies a given state predicate Safe

Why NNCS Verification?

NNCS Used in Safety-Critical Applications

Autonomous Cars

Industrial Control

Healthcare

Our Approach: Key Difference with Prior Work

Majority of Past Work	Bounded-Time Horizon	Reachability Analysis
Ours	Infinite-time Horizon	Inductive Invariant Method

Inductive Invariant Method

Discover a state predicate $IndInv$, such that

(1) $Init \implies IndInv$

(2) $(IndInv \land Next) \implies IndInv'$

(3) $IndInv \implies Safe$

Challenges for Inductive Invariant Method

Discovery of $IndInv$

Check if candidate $IndInv$ is indeed inductive

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This paper

Inductiveness Check in NNCS

Inductiveness

$$(IndInv \wedge Next) \implies IndInv'$$

NNCS

$$Next = Next_{NNC} \wedge Next_{ENV}$$

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Inductiveness Check in NNCS

Inductiveness

$$(IndInv \wedge Next) \implies IndInv'$$

NNCS

$$Next = Next_{NNC} \wedge Next_{ENV}$$

Inductiveness in NNCS

Monolithic Method:
Check this formula directly

$$(IndInv \wedge Next_{NNC} \wedge Next_{ENV}) \implies IndInv'$$

Monolithic Inductiveness Check

Monolithic Inductiveness Check

$$(IndInv \land Next_{NNC} \land Next_{ENV}) \implies IndInv'$$

Monolithic Inductiveness Check

$$(IndInv \wedge Next_{NNC} \wedge Next_{ENV}) \implies IndInv'$$

SMT Solvers
(e.g. Z3)
Do not scale due to the size of NN controller

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NN Verifier

(e.g. α, β-CROWN)

Limited expressiveness of verifiable specifications

Monolithic Inductiveness Check

$$(IndInv \wedge Next_{NNC} \wedge Next_{ENV}) \implies IndInv'$$

SMT Solvers

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Do not scale due to the size of NN controller

NN Verifier

(e.g. α, β-CROWN)

Limited expressiveness of verifiable specifications

Monolithic method cannot efficiently check if the candidate $IndInv$ is indeed inductive

Our Contribution

Compositional Method

Replace **monolithic** check

$$(M) \qquad (IndInv \land Next_{NNC} \land Next_{ENV}) \implies IndInv'$$

with 2 **easier** checks

$$(C1) \qquad (IndInv \land Next_{NNC}) \implies Bridge$$

$$(C2) \qquad (Bridge \land Next_{ENV}) \implies IndInv'$$

Compositional Method

Replace **monolithic** check

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SMT Solvers can handle this (but we help with parallelization)

Compositional Method

Soundness and Completeness

$$
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Compositional Method

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Theorem: Soundness

If $(C1)$ and $(C2)$ hold, then (M) holds

Compositional Method

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Theorem: Soundness

If $(C1)$ and $(C2)$ hold, then (M) holds

Theorem: Completeness

If (M) holds, then there exists $Bridge$ such that $(C1)$ and $(C2)$ hold

Compositional Method

Questions

$$(C1) \qquad (IndInv \land Next_{NNC}) \implies Bridge$$

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Compositional Method

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How to **discover** $Bridge$

Compositional Method

Questions

$$(C1) \qquad (IndInv \land Next_{NNC}) \implies Bridge$$
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How to **discover** $Bridge$

How to **check** $(C1)$ and $(C2)$ in a scalable way

Compositional Method

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How to **discover** $Bridge$

Automatic method

How to **check** $(C1)$ and $(C2)$ in a scalable way

Automatic $Bridge$ Construction

Decompose $IndInv$

$$IndInv \iff (p_1 \lor p_2 \lor \cdots \lor p_n)$$

Automatic *Bridge* Construction

Decompose $IndInv$

$$IndInv \iff (p_1 \vee p_2 \vee \cdots \vee p_n)$$

For each state predicate p_i, compute postcondition ψ_i, such that

$$(p_i \wedge Next_{NNC}) \implies \psi_i$$

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NN controller input constraint

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$$Bridge := (p_1 \wedge \psi_1) \vee \cdots \vee (p_n \wedge \psi_n)$$

Condition $(C1)$ Holds by Construction!

$$(IndInv \land Next_{NNC}) \implies Bridge$$

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Replaced by $IndInv$ decomposition

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$Bridge := (p_1 \land \psi_1) \lor \cdots \lor (p_n \land \psi_n)$

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$$\bigvee_i (p_i \land Next_{NNC}) \implies \bigvee_i (p_i \land \psi_i)$$

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This holds because the NN verifier guarantees

$$(p_i \land Next_{NNC}) \implies \psi_i$$

How to Check Condition $(C2)$ Efficiently?

$$(Bridge \land Next_{ENV}) \implies IndInv'$$

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This conjunction of smaller formulas can be checked in parallel!

Algorithm

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Decompose $IndInv$ into
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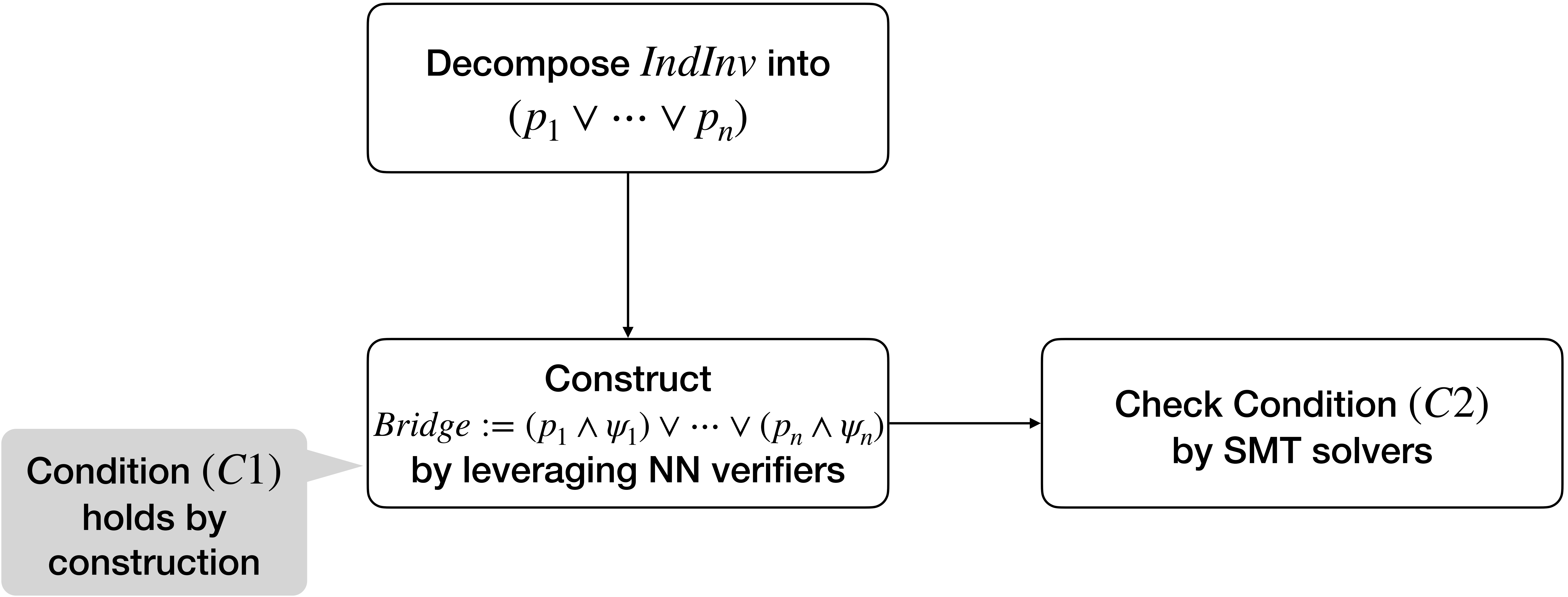
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Construct $Bridge := (p_1 \wedge \psi_1) \vee \cdots \vee (p_n \wedge \psi_n)$ by leveraging NN verifiers
Check Condition $(C2)$ by SMT solvers
Condition $(C1)$ holds by construction

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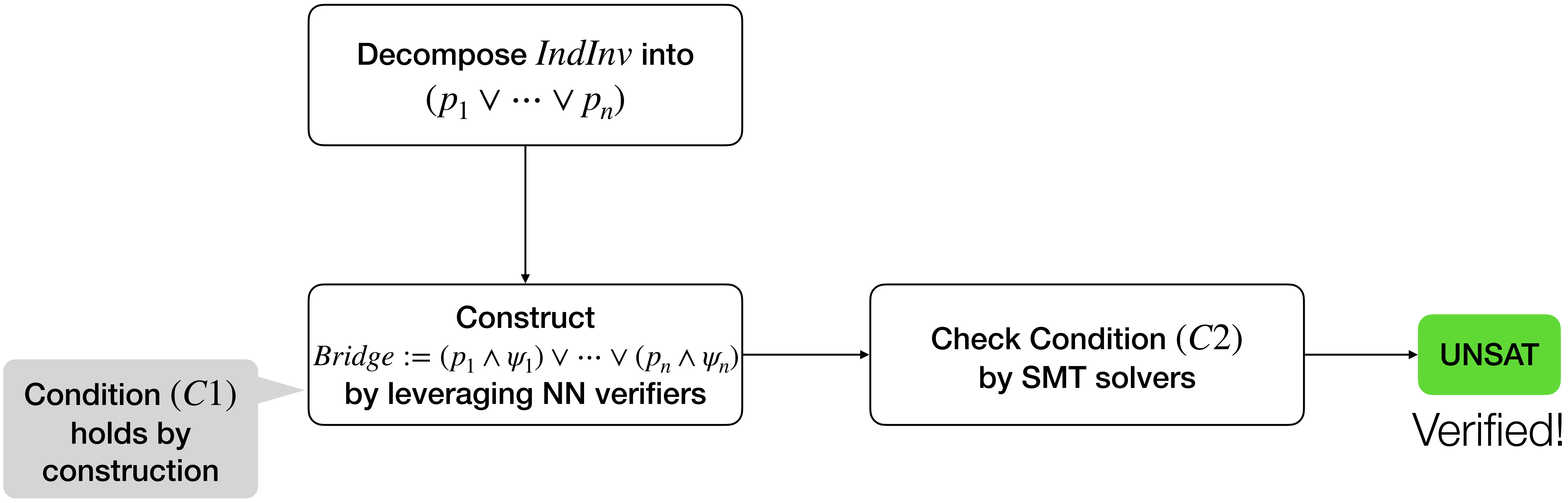
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UNSAT
Verified!

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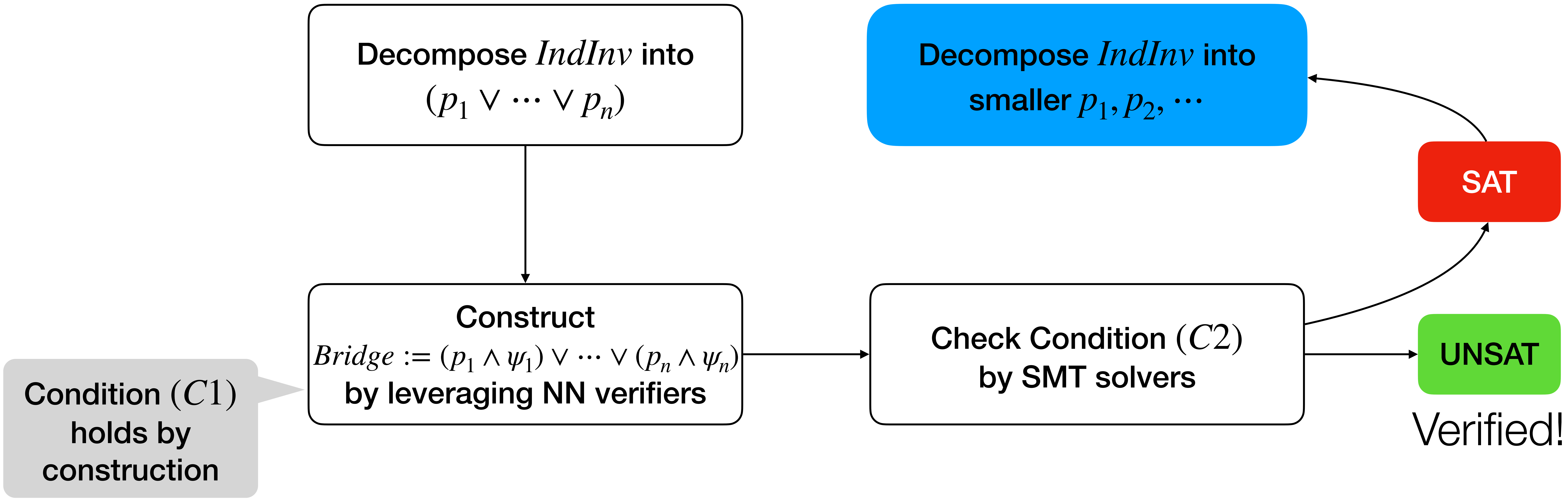

Decompose $IndInv$ into $(p_1 \vee \cdots \vee p_n)$
Decompose $IndInv$ into smaller $p_1, p_2, \cdots$
SAT
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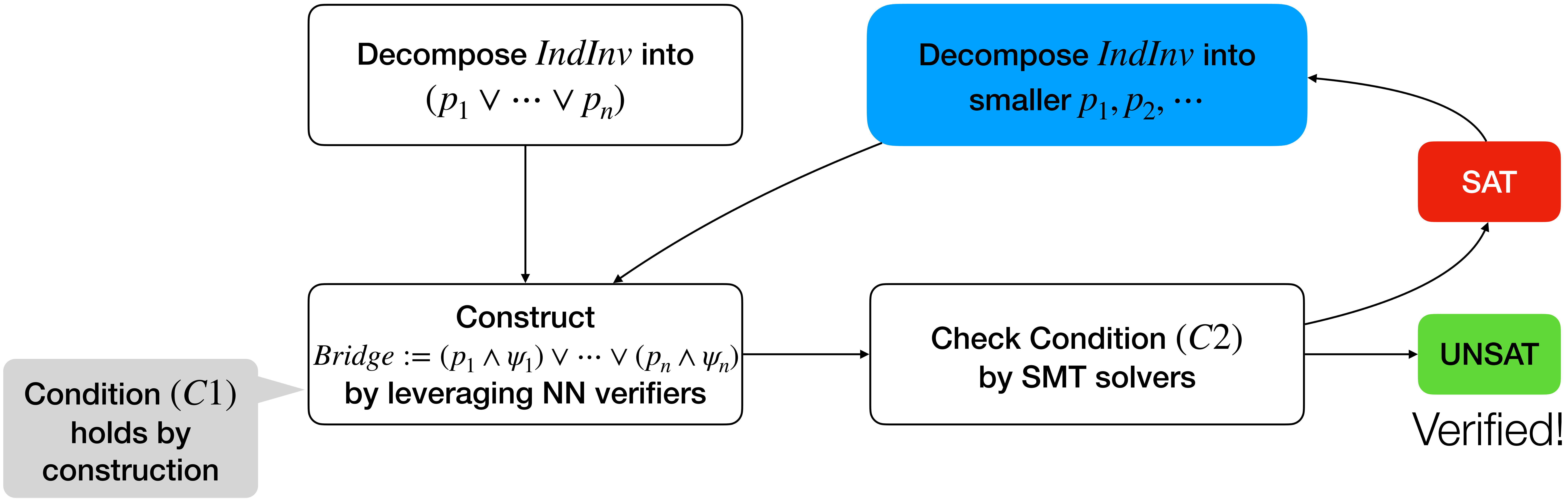

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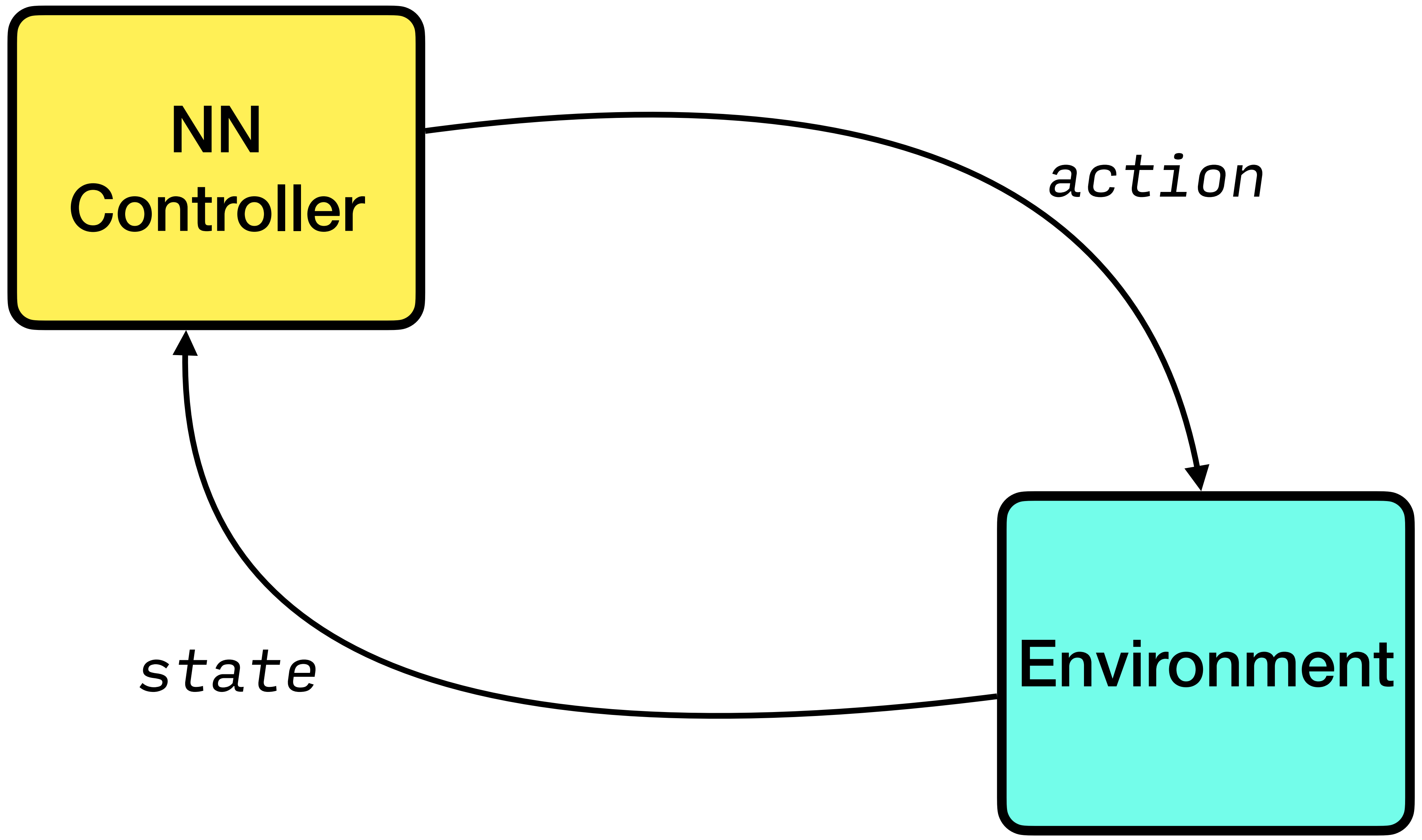
2D Navigation Case Study



State variables

$$x, y \in \mathbb{R}$$

NN controller

$$(a, b) = NN(x, y)$$

Environment

$$0.5 \leq c \leq 1.0$$
$$x' = x + 0.1 \cdot c \cdot a,$$
$$y' = y + 0.1 \cdot c \cdot b$$

Evaluation

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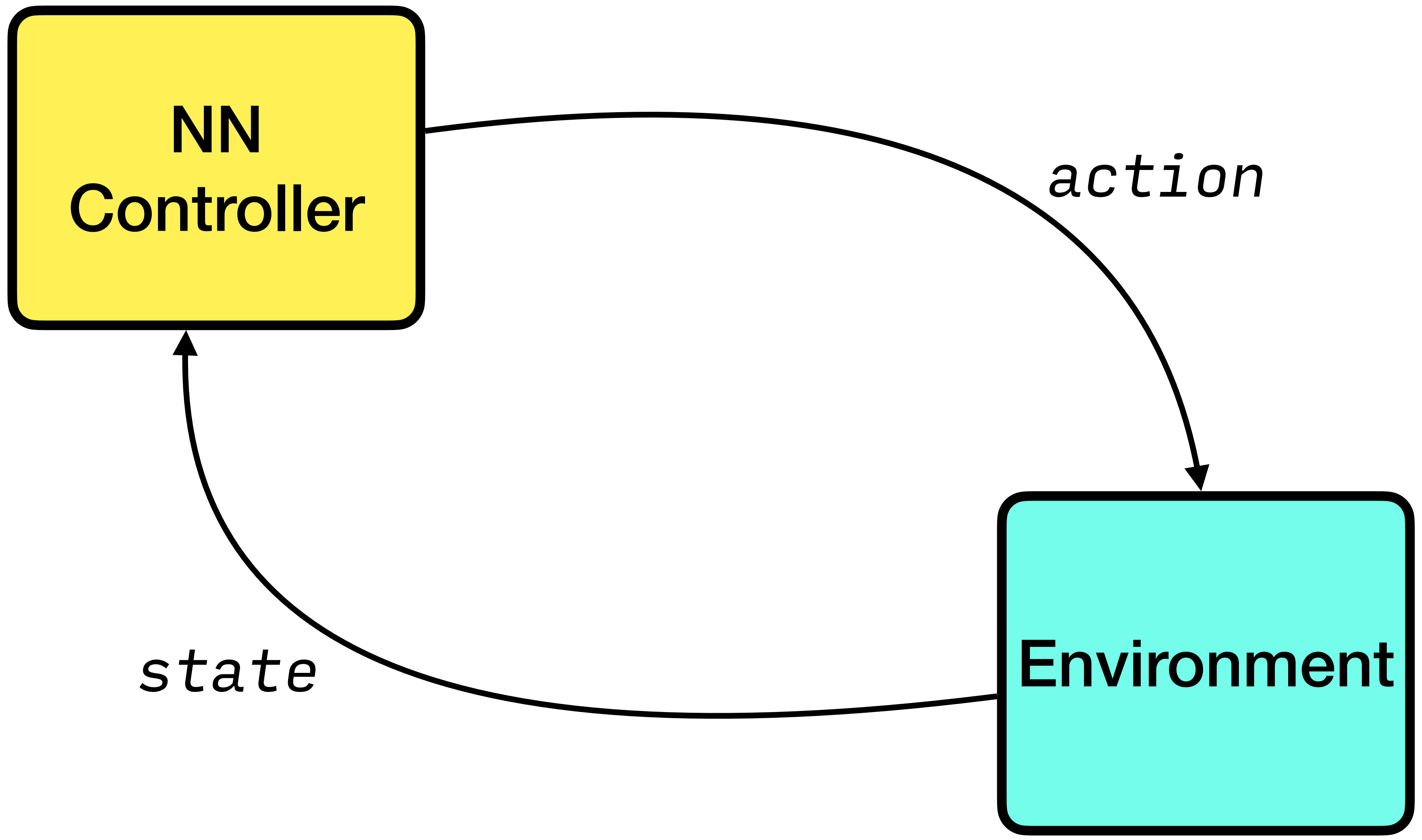

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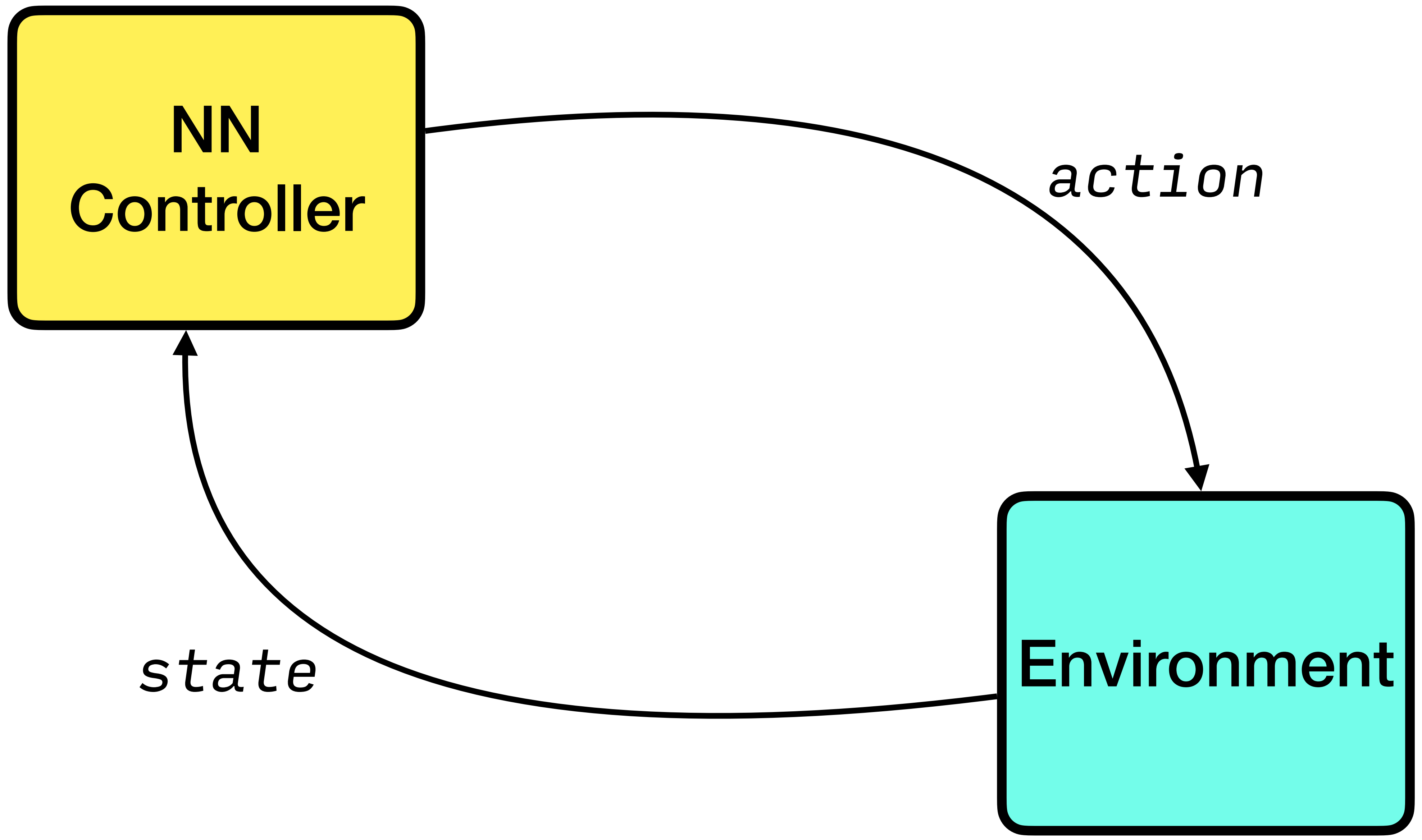

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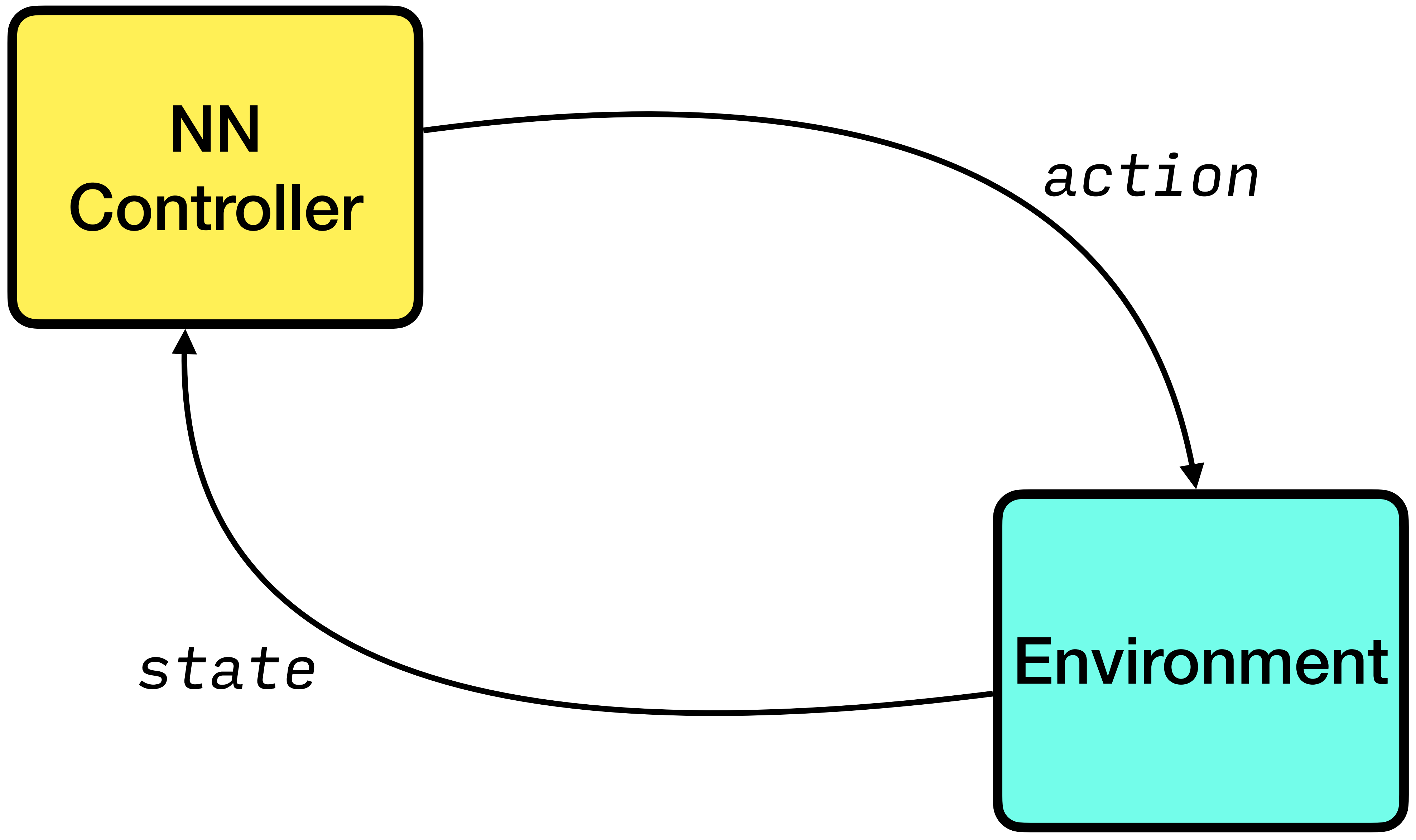

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Non-deterministic, discrete-time

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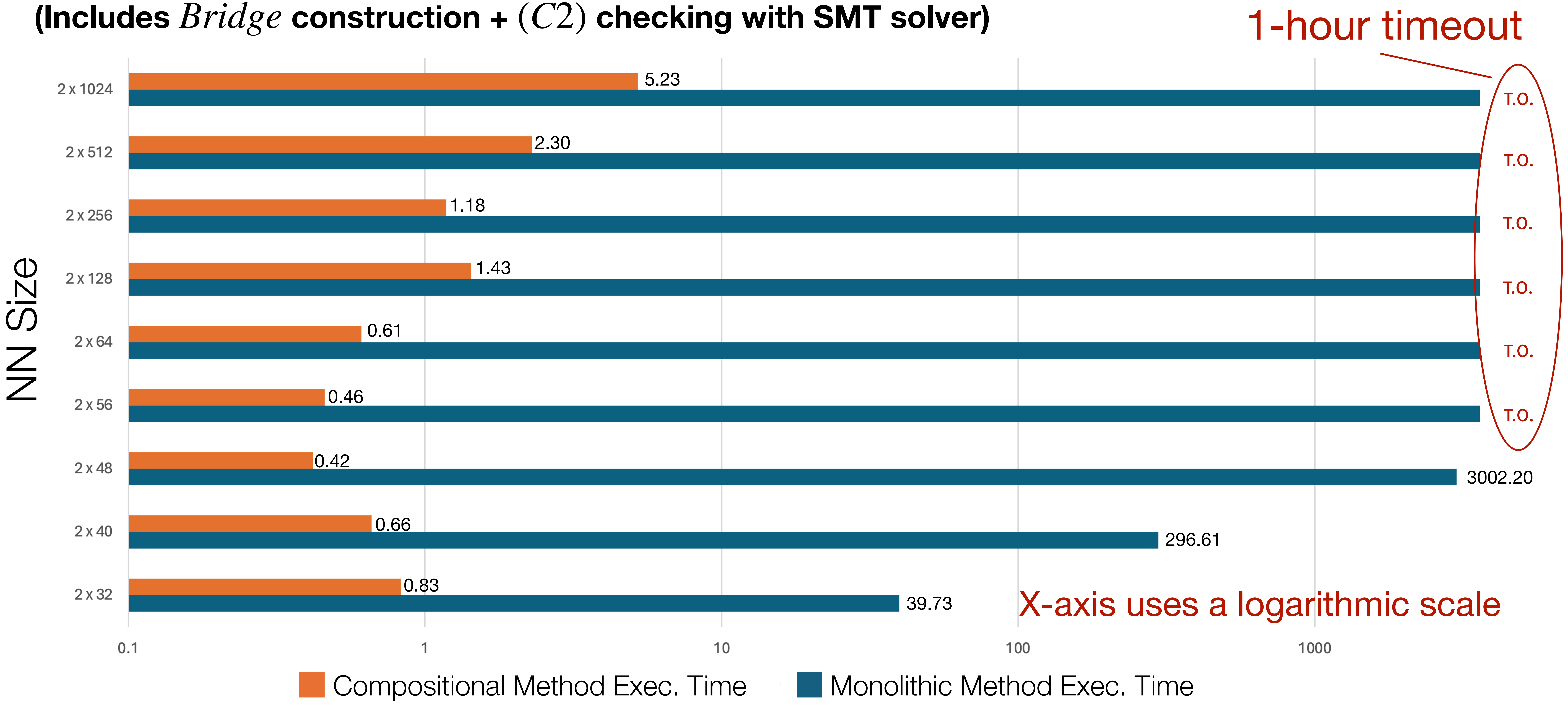
- Z3 and AutoLIRPA for the compositional method

Execution Times (seconds)

(Includes $Bridge$ construction + $(C2)$ checking with SMT solver)



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- AutoLIRPA on GPU with CUDA enabled

Evaluation

2D Maze Case Study

NN size	Verified?		Monolithic execution time		Compositional execution time		#Splits		#SMT queries		#NNV queries	
	Det	NDet	Det	NDet	Det	NDet	Det	NDet	Det	NDet	Det	NDet
2×32	T	T	51.59	39.73	0.73	0.83	12	14	61	71	49	57
2×40	T	T	113.69	296.61	0.70	0.66	13	13	66	66	53	53
2×48	T	T	410.14	3002.20	0.52	0.42	10	8	51	41	41	33
2×56	T	T	1203.76	T.O.	0.42	0.46	8	9	41	46	33	37
2×64	T	T	T.O.	T.O.	0.76	0.61	15	12	76	61	61	49
2×128	F	T	T.O.	T.O.	2.21	1.43	64	28	225	141	160	113
2×256	T	T	T.O.	T.O.	1.68	1.18	27	23	136	116	109	93
2×512	T	T	T.O.	T.O.	3.04	2.30	60	45	301	226	241	181
2×1024	T	T	T.O.	T.O.	1.94	5.23	38	102	191	511	153	409

Falsification

Falsification

Theorem: Completeness

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Theorem: Completeness

$\boxed{\text{If } (M) \text{ holds,}}$ then there exists $Bridge$ such that $(C1)$ and $(C2)$ hold

If (M) does not hold, we want to avoid hopeless search for $Bridge$

Falsification

Theorem: Completeness

If (M) holds, then there exists Bridge such that $(C1)$ and $(C2)$ hold

If (M) does not hold, we want to avoid hopeless search for Bridge

A practical **falsification** heuristic inherits the decomposition ideas

Algorithm with Falsification

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Decompose $IndInv$ into
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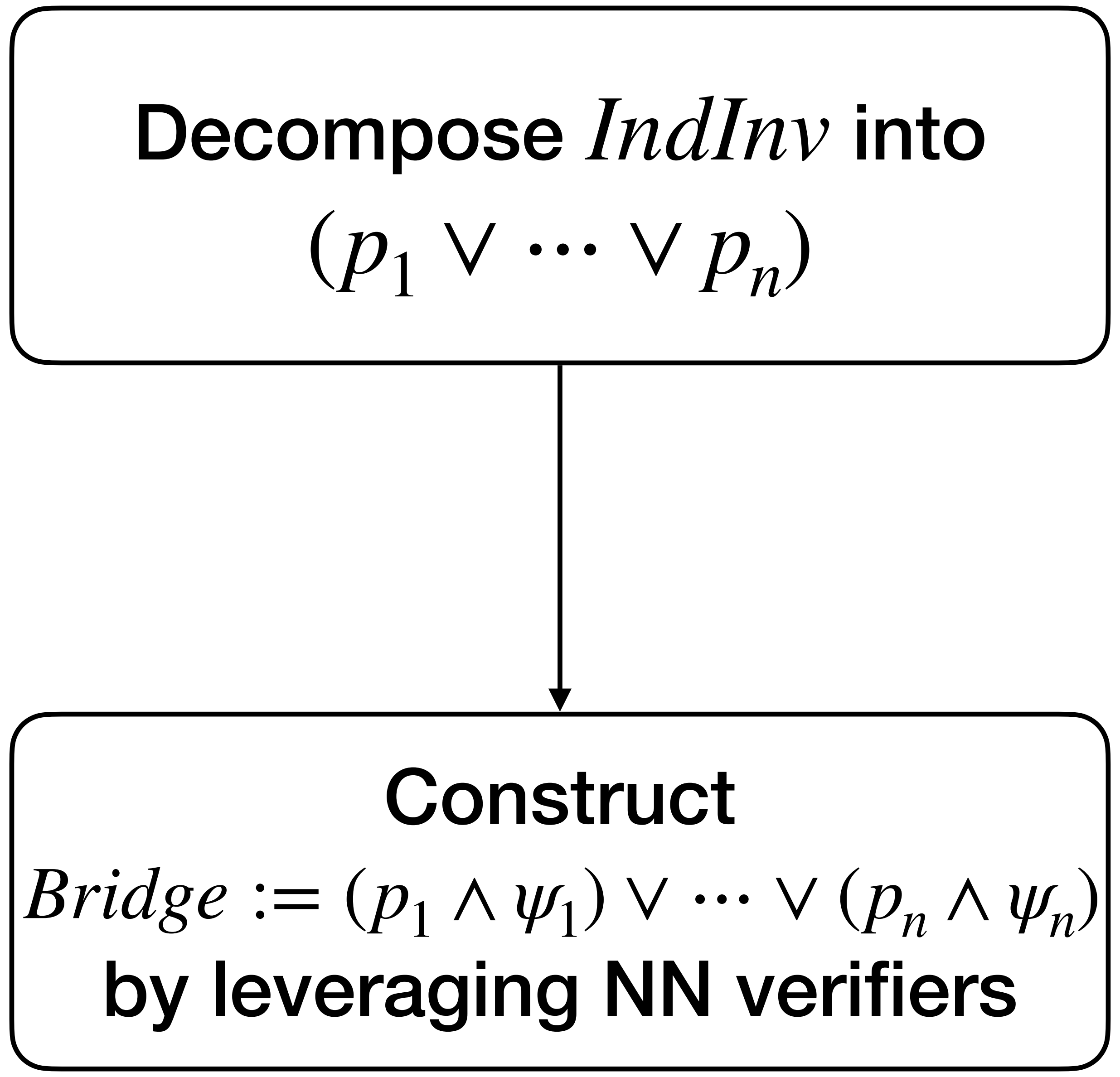

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Condition $(C1)$
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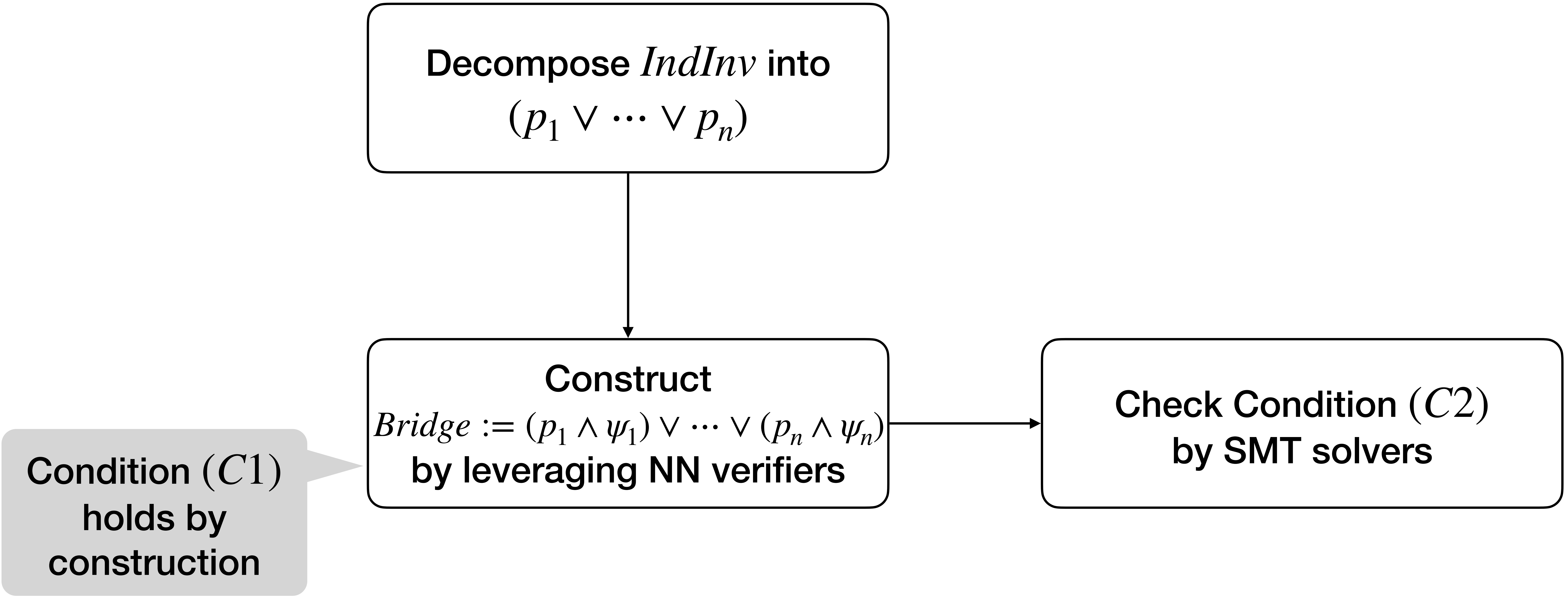

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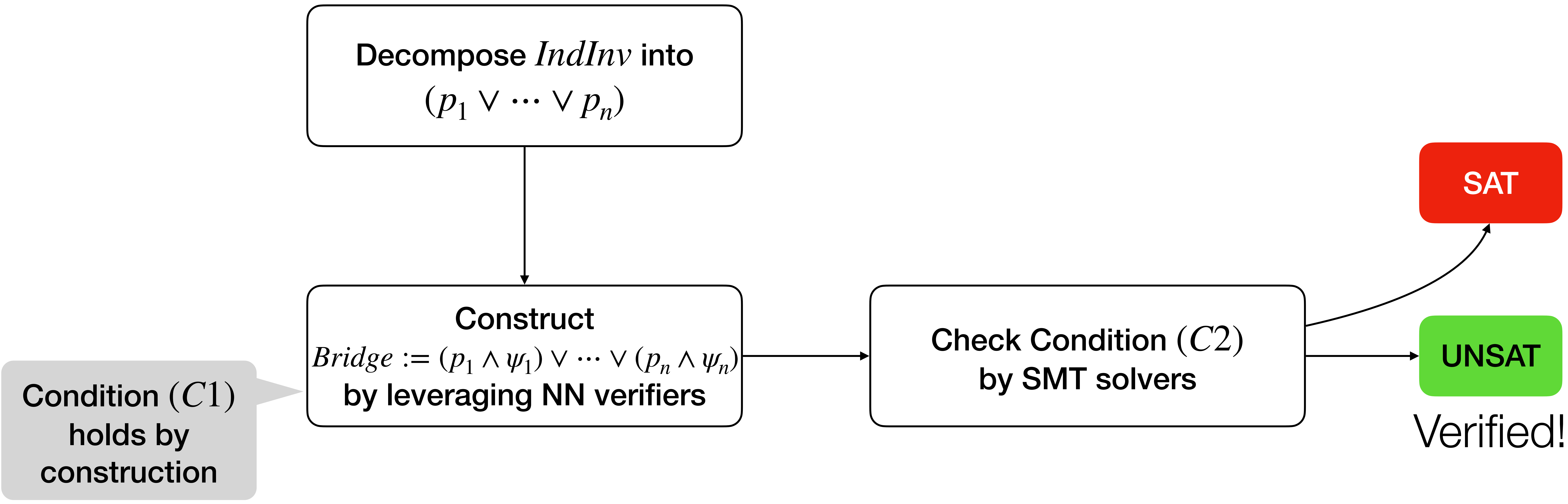

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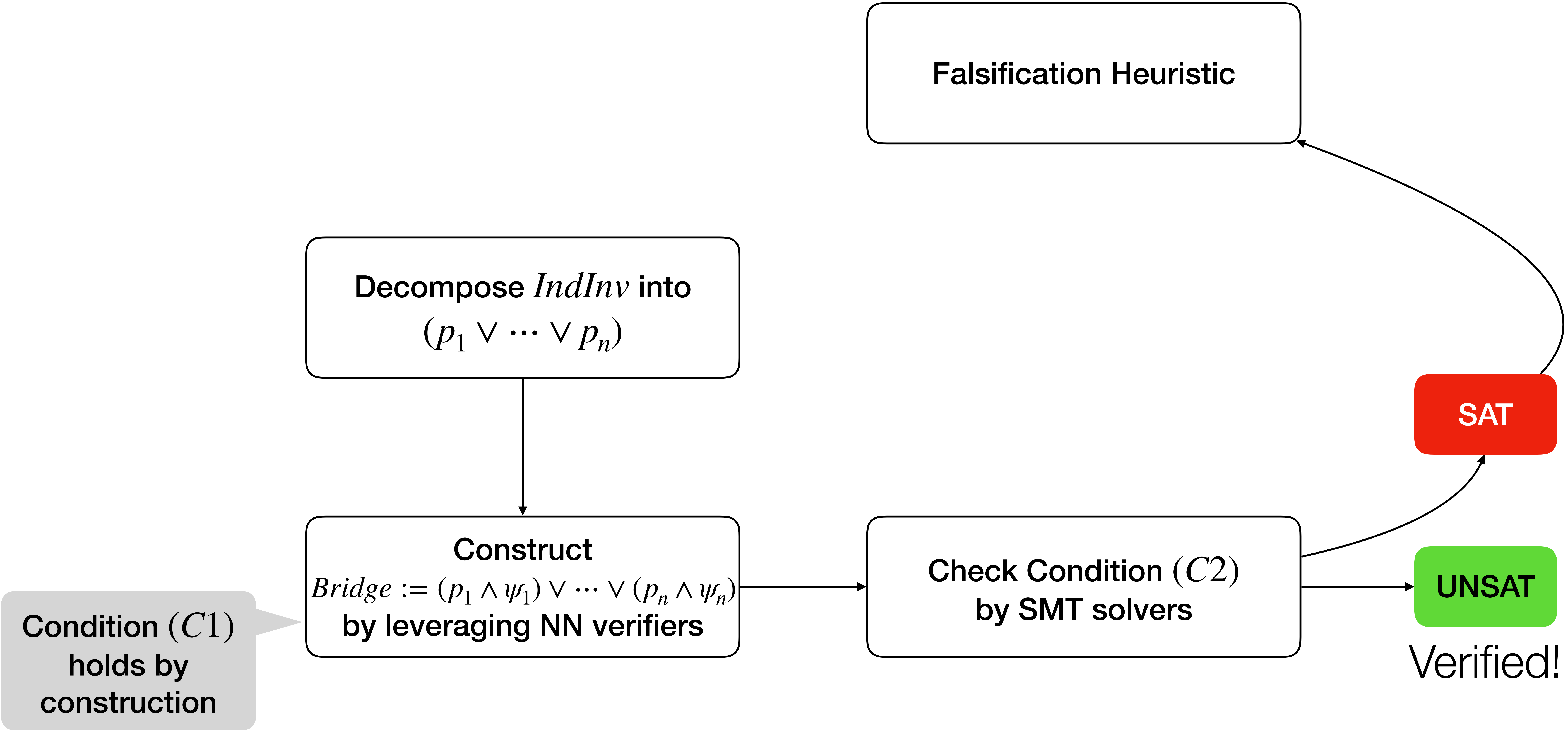

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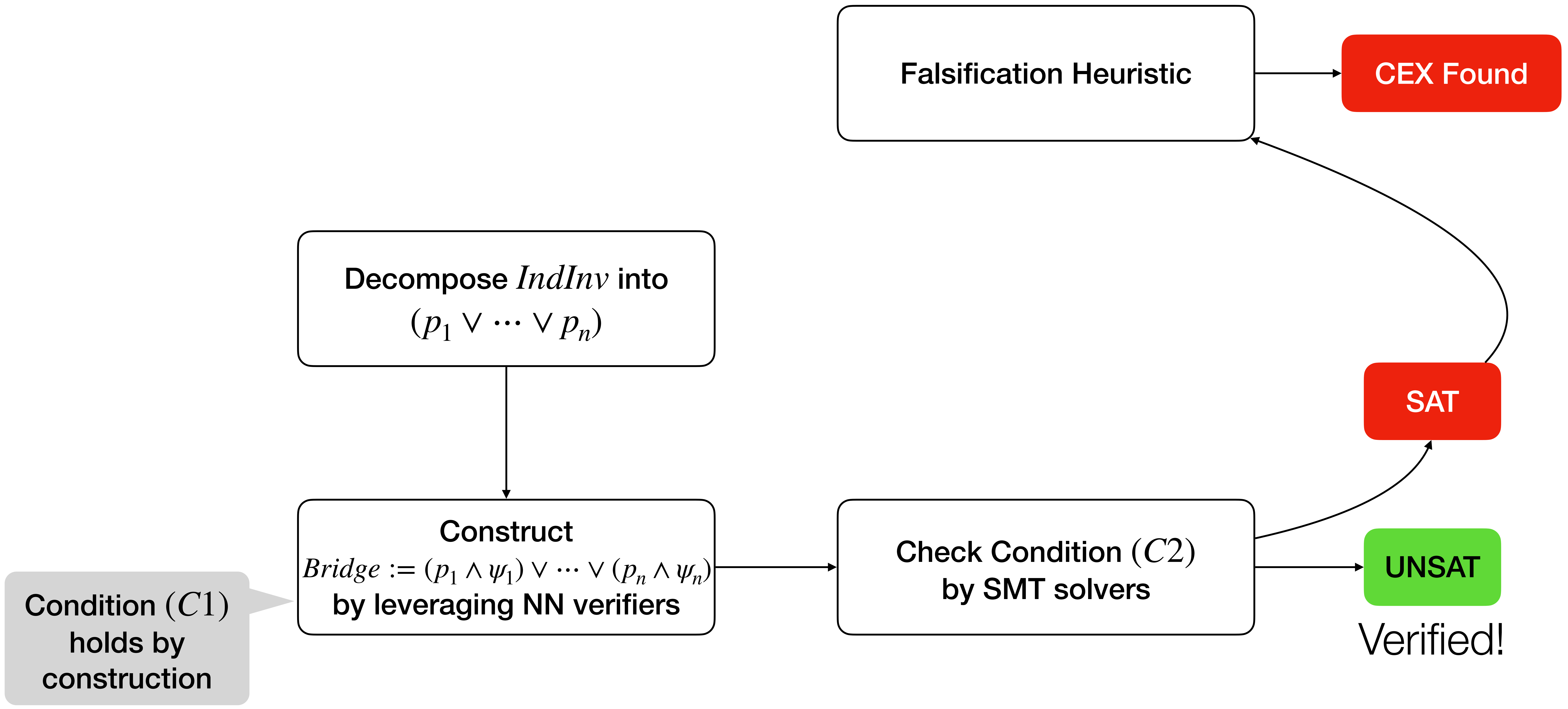
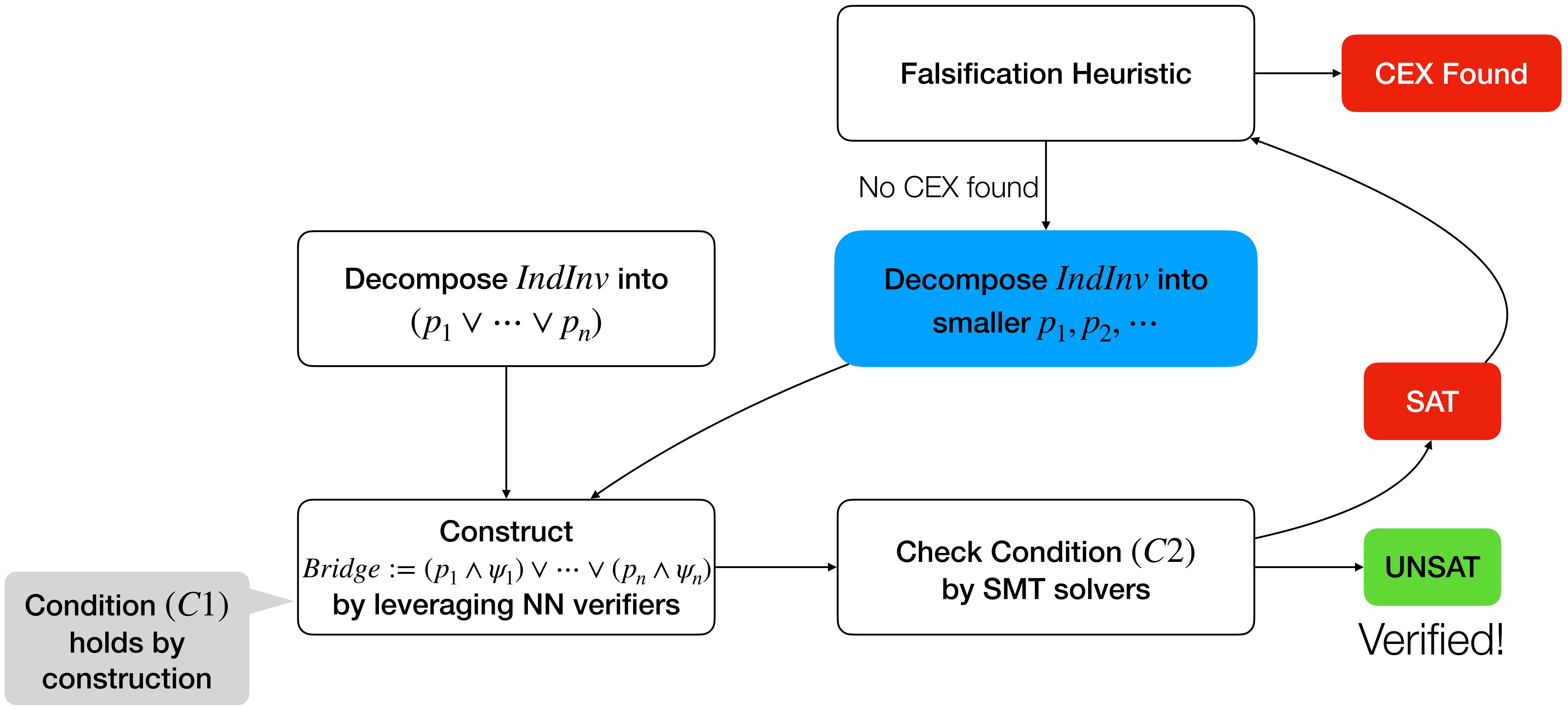


Algorithm with Falsification



Termination of the Algorithm

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Theorem: Completeness

If (M) holds, then there exists $Bridge$ such that $(C1)$ and $(C2)$ hold

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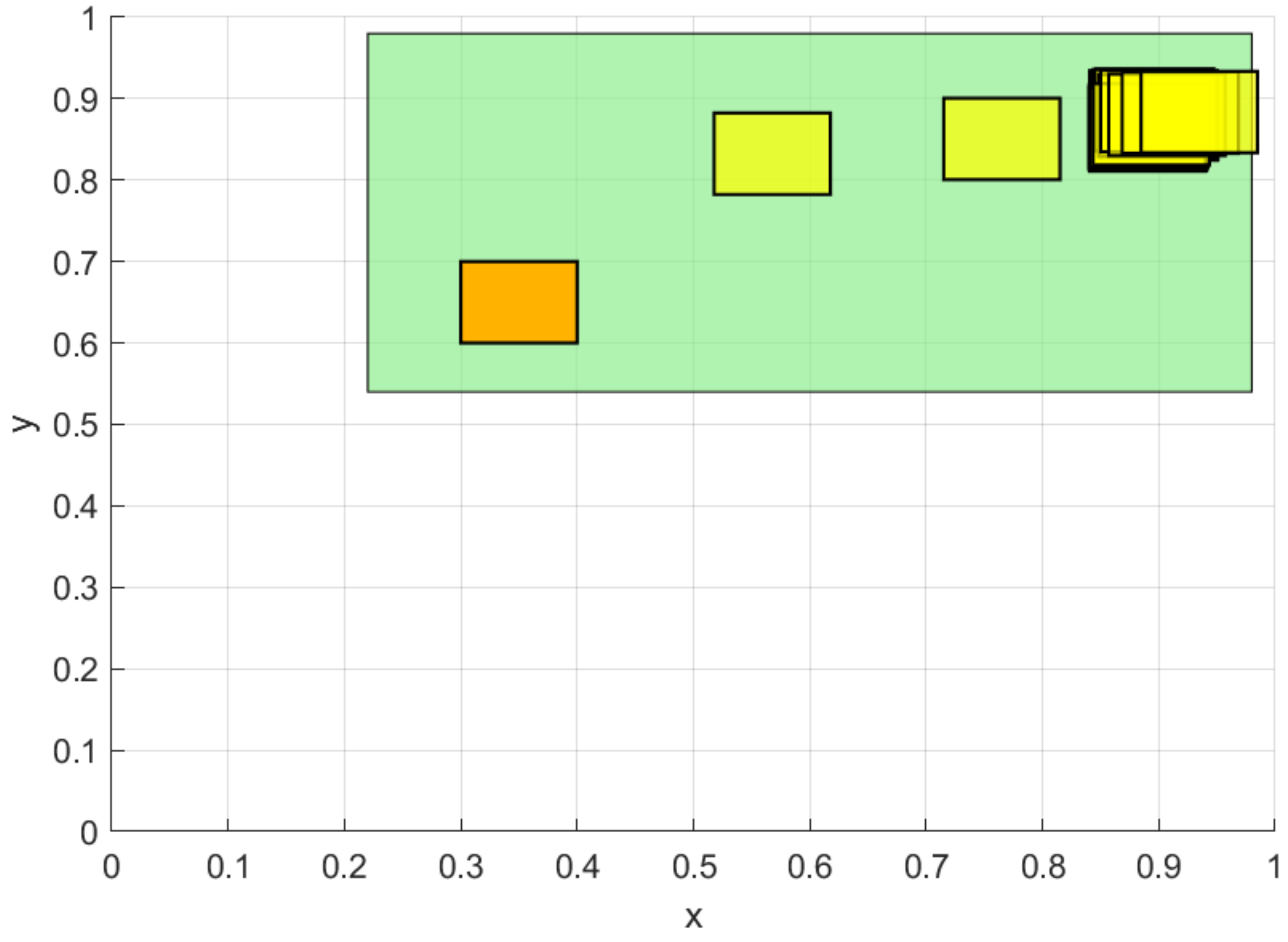
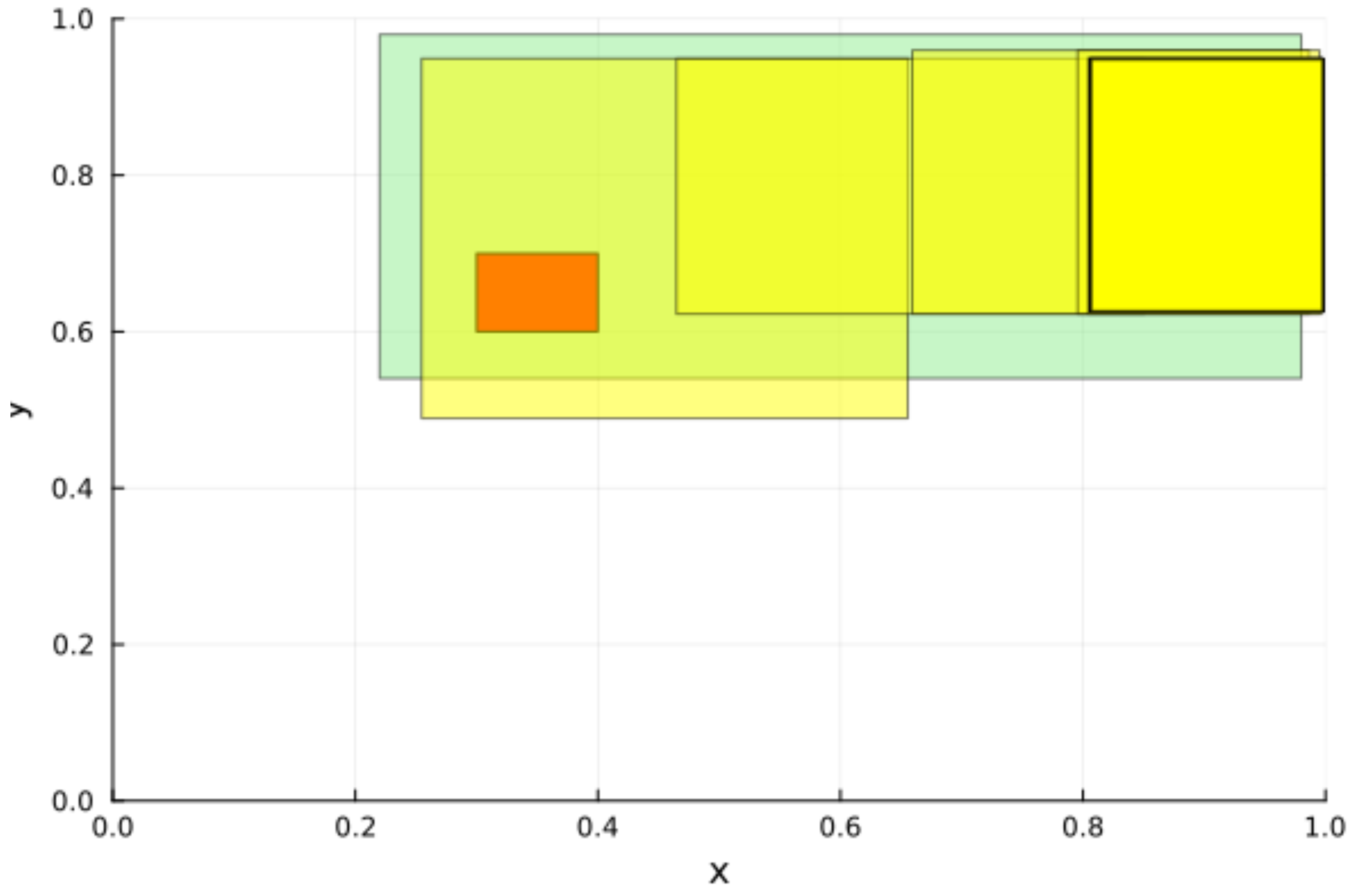
Algorithm

Termination not guaranteed

Compare with Reachability Analysis



NNV (2 x 512)



JuliaReach (2 x 512)

Why NN Verifiers Not Working for Monolithic Method

I. Encode $Next_{ENV}$ into NNV Specification

$$(IndInv \wedge Next_{NNC} \wedge Next_{ENV}) \implies IndInv'$$

Use NNV as black-box

NN takes x, outputs y

NNV Specification

Given $P(x)$, guarantee $Q(y)$

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not straightforward to encode,
e.g. our case study

Use NNV as black-box

NNV Specification

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$Next_{ENV}$ may be non-deterministic, while NNVs typically assume NN to be deterministic