CS1800
Discrete Structures
Fall 2017

Lecture 24
11/1/17

Last time

- Expectation

Today

- Applications of Expectation
  - Variance & Standard Deviation
  - Entropy

Next time

- Conditional Prob.
- Bayes Theorem

Variance

Consider heights

case 1	case 2	case 3
4'10'' 5' 5'2''	4' 5' 6'	3' 5' 7'
$E\{X_1\} = 5'$	$E\{X_2\} = 5'$	$E\{X_3\} = 5'$

How to measure "variability"

3 ways

① $Y = X - E\{X\}$ ✗ neg. & pos. cancel

② $Y = |X - E\{X\}|$ - mean absolute deviation

③ $Y = (X - E\{X\})^2$ - variance

case 2

① $$E[Y] = \sum_{\omega \in \Omega} Y(\omega) \cdot p(\omega)$$
$$= (-12'') \cdot 1/3 + (0'') \cdot 1/3 + (+12'') \cdot 1/3$$
$$= 0$$

② $$E\{Y\} = |-12''| \cdot 1/3 + |0''| \cdot 1/3 + |+12''| \cdot 1/3$$
$$= 12 \cdot 1/3 + 0 \cdot 1/3 + 12 \cdot 1/3 = 8 \text{ inches}$$

case 1

③ $$E\{Y\} = E[(X - E\{X\})^2] = (-2'')^2 \cdot 1/3 + (0'')^2 \cdot 1/3 + (+2'')^2 \cdot 1/3$$
$$= 4/3 + 0 + 4/3 = 8/3 \text{ inches}^2$$

variance σ^2

Take square root, get back inches...

$$\sigma^2 = Var(X) = E\{(X - E\{X\})^2\}$$

$$\Rightarrow \quad \sigma = \sqrt{Var(X)} = \sqrt{E[(X - E\{X\})^2]}$$ – back in units originally measured.

e.g.

case 1 : $\sigma^2 = Var(X) = 8/3 \text{ inches}^2$

$$\sigma = \sqrt{Var(X)} = \sqrt{8/3 \text{ inches}^2} = 1.63''$$

case 3 : ... $\sigma = 19.6''$

Exam $\quad E\{X\} = 76$

$\sigma = 15 \qquad \pm 2/3\,\sigma$ – quartiles

$2/3\,\sigma = 10$ pts

Entropy

- Consider 8 letters only $\{A, B, C, D, E, F, G, H\}$
- Need 3-bits

$$\left.\begin{array}{l} A \to 000 \\ B \to 001 \\ \vdots \\ H \to 111 \end{array}\right\} \text{fixed length code}$$

- suppose must encode n-things, need: $\lceil \log_2 n \rceil$
- Efficiency of code is measured in bits-per-character on average, BPC. $BPC = 3$
- variable length code: assign short codes to frequent letters
  long codes to infrequent letters.

$$BPC = \sum_{w \in \Omega} X(w) \cdot p(w) = \sum_i l_i \cdot p_i$$

l_i: code length for i^{th} letter

p_i: probability of i^{th} letter

Why must we have long codes to make up for short codes?

Analog to PHP: Kraft's Inequality

Example:

A	B	C	D	E	F	G	H
00	01	010	011	100	101	110	111

BBB → 010101

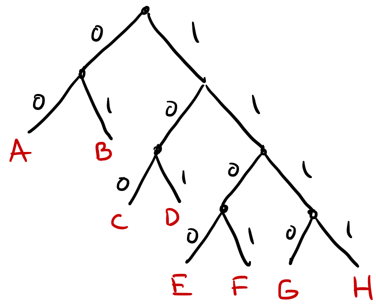
CF → 010101

Suppose we have following skewed distribution

$$\begin{array}{ccccccccc} & A & B & C & D & E & F & G & H \\ P_1 = (& 1/4, & 1/4, & 1/8, & 1/8, & 1/16, & 1/16, & 1/16, & 1/16 \,) \end{array}$$

$$(00,\ 01,\ 100,\ 101,\ 1100,\ 1101,\ 1110,\ 1111)$$

Code Tree:



ADF → 001011101

$$\underset{A}{00}\,|\,\underset{D}{101}\,|\,\underset{F}{1101}$$

$$BPC = \sum_i l_i \cdot p_i$$

$$= 2 \cdot 1/4 + 2 \cdot 1/4 + 3 \cdot 1/8 + 3 \cdot 1/8 + 4 \cdot 1/16 + \cdots + 4 \cdot 1/16$$

$$= 2.75$$

* 8.33% savings over fixed code of length 3.