CS1800
Discrete Structures
Fall 2017

Lecture 19
10/19/17

Last time

- Finish Perm. & Comb.
  - examples
  - balls-in-bins

Today

- Binomial Theorem

Next time

- Counting Problem Examples

Binomial theorem: $(x+y)^n$

$(x+y)^0 = 1$

$(x+y)^1 = x + y$

$(x+y)^2 = x^2 + 2xy + y^2 = (x+y)(x+y)$

choose x or y choose x or y

	x	y
x	x^2	xy
y	yx	y^2

$(x+y)^3 = (x+y)(x+y)(x+y)$

0 y's ←	F	x	x	→	x^2
1 y ←	O	x	y	→	xy
	I	y	x	→	yx
2 y's ←	L	y	y	→	y^2

$\Rightarrow x^2 + 2xy + y^2$

Q: How many y's do you choose → dictates term, e.g., xy^2 vs. x^2y

How many ways to do so? → dictates coefficient in front of term.

A:

	# ways	term
0 y's	1	x^3
1 y	$3 = \binom{3}{1}$	x^2y
2 y's	$3 = \binom{3}{2}$	xy^2
3 y's	1	y^3

$\Rightarrow$

1   3   3   1

$x^3 + 3x^2y + 3xy^2 + y^3$

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

How many y's?

	# ways	term
0	$1 = \binom{4}{0}$	x^4
1	$4 = \binom{4}{1}$	$x^3 y$
2	$6 = \binom{4}{2}$	$x^2 y^2$
3	$4 = \binom{4}{3}$	$x y^3$
4	$1 = \binom{4}{4}$	y^4

1 4 6 4 1

$$\Rightarrow x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^n = (x+y)(x+y) \cdots (x+y) \leftarrow n \text{ terms}$$

# y's	# ways	term
0	$\binom{n}{0}$	x^n
1	$\binom{n}{1}$	$x^{n-1} y$
2	$\binom{n}{2}$	$x^{n-2} y^2$
$\vdots$	$\vdots$	$\vdots$
n	$\binom{n}{n}$	y^n

$$\Rightarrow \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} y^n$$

$$= \boxed{\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j}$$

Application:

$$(x+2y^{-2})^6$$

Q: What is the term that includes y^{-8}?

$$= \overbrace{(x+2y^{-2})(x+2y^{-2})\cdots(x+2y^{-2})}^{6}$$

$\Rightarrow$ to get y^{-8}, need to expand by $2y^{-2}$ 4 times

$$\binom{6}{4} x^2 (2y^{-2})^4$$

$$= \binom{6}{2} x^2 (2y^{-2})^4$$

$$= \frac{6\cdot 5}{\cancel{2}\cdot 1} x^2 \cdot 2^{\cancel{4}\,3} y^{-8}$$

$$= 240\, x^2 y^{-8} \quad \checkmark$$

$$(x+y)^4 = (x+y)(x+y)^3$$

$$= (x+y)(x^3+3x^2y+3xy^2+y^3)$$

$$\begin{array}{rcccccccccc} = & x^4 & + & 3x^3y & + & 3x^2y^2 & + & xy^3 & & \\ + & & & x^3y & + & 3x^2y^2 & + & 3xy^3 & + & y^4 \\ \hline & x^4 & + & 4x^3y & + & 6x^2y^2 & + & 4xy^3 & + & y^4 \end{array}$$

$$\begin{array}{ccccc} 1 & 3 & 3 & 1 & \\ & 1 & 3 & 3 & 1 \\ \hline 1 & 4 & 6 & 4 & 1 \end{array}$$

$\binom{3}{2}$ $\binom{3}{1}$ $\binom{4}{2}$

$$\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$$

need 2 y's

expand by x in first term; need 2 y's from 2nd term

expand by y in first term, need 1 y in 2nd term

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$$

Choose 3 out of 8 people

$$\binom{8}{3} = \binom{7}{3} + \binom{7}{2}$$

Pascal's Triangle

$$(x+y)^n$$

$j = \#\, y$'s

	0	1	2	3	4 ...
$n=0$	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1
5					
⋮					

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Applications & Consequences

① What is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

Consider $S = \{a, b, c\}$

$P(S)$	0	$\emptyset$	$1 = \binom{3}{0}$
	1	$\{a\}$ $\{b\}$ $\{c\}$	$3 = \binom{3}{1}$
	2	$\{a,b\}$ $\{b,c\}$ $\{a,c\}$	$3 = \binom{3}{2}$
	3	$\{a,b,c\}$	$1 = \binom{3}{3}$
			8

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^{j}$$

1 1

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot 1^{n-j} \cdot 1^{j} = \sum_{j=0}^{n} \binom{n}{j}$$

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1331$$
$$11^4 = 14641$$

$$11^n = (\overset{x}{1} + \overset{y}{10})^n = \sum_{j=0}^{n} \binom{n}{j} \cdot 1^{n-j} \cdot 10^j$$
$$= \sum_{j=0}^{n} \binom{n}{j} \cdot 10^j$$
$$= \binom{n}{n} \cdot 10^n + \binom{n}{n-1} 10^{n-1} \cdots \binom{n}{0} \cdot 10$$

$$11^3 = (1+10)^3 = \binom{3}{3} \cdot 10^3 + \binom{3}{2} 10^2 + \binom{3}{1} \cdot 10^1 + \binom{3}{0} \cdot 10^0$$
$$= 1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10 + 1$$
$$= 1331$$

1

1 1

1 2 1

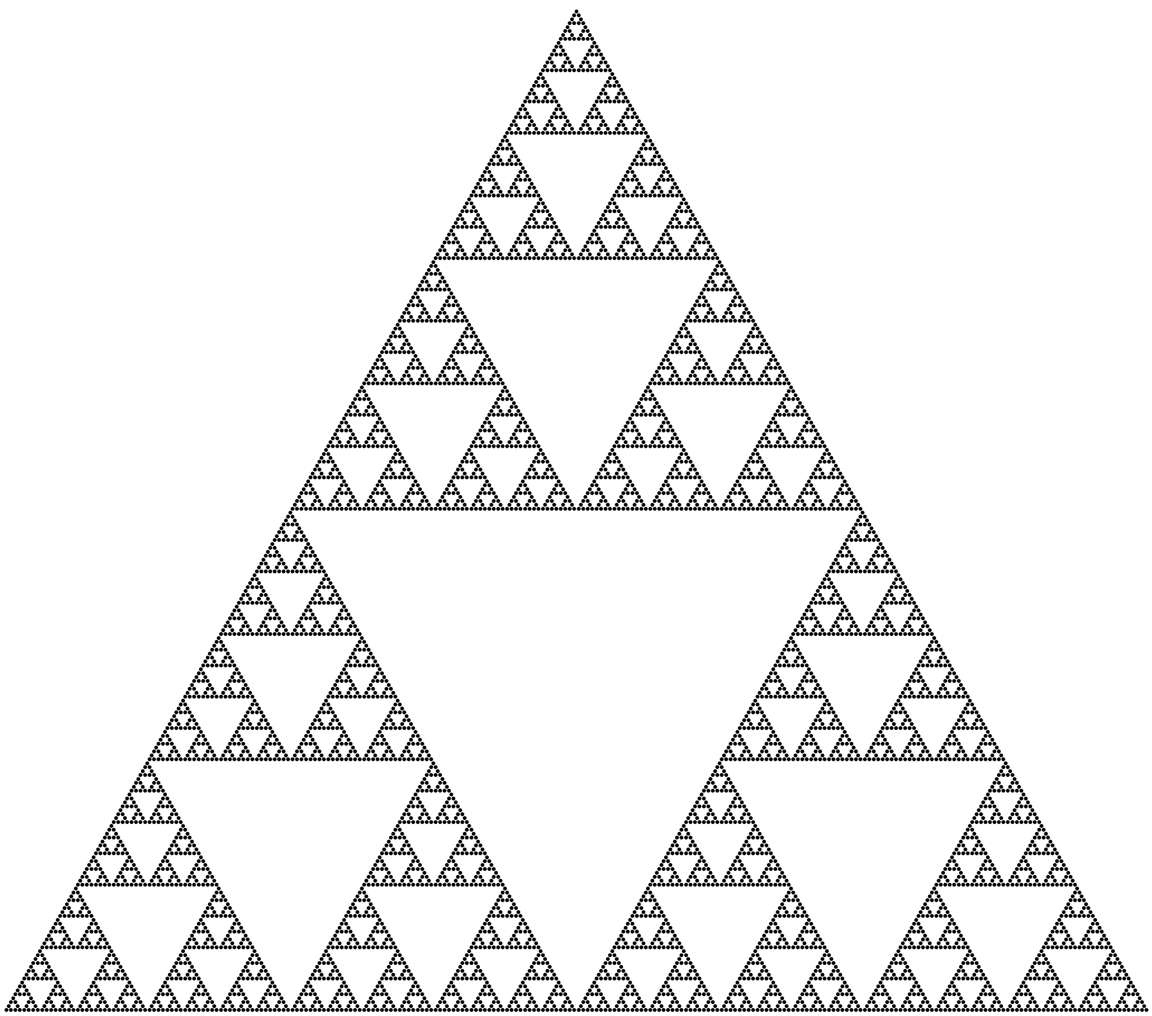
1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Discrete Structures



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