CS1800
Discrete Structures
Fall 2017

Lecture 17
10/16/17

Last time

- Finish sets & Set operations
- Basic rules for counting
  - product rule
  - sum rule
    - principle of inclusion-exclusion

Today

- Pigeonhole Principle
- Permutations & Combinations

Next time

- Balls-in-bins
- Binomial Theorem

Announcement

- Lecture Videos
- Lecture Notes

Pigeonhole Principle

If place $n+1$ (or more) objects in n boxes, then at least one box has 2 or more objects.

Claim: Every integer n has a multiple that contains only 1's & 0's in decimal.

e.g.
$2 \to 10$
$4 \to 100$
$5 \to 10$
$3 \to ?$

e.g. 3

$$4\left\{\begin{array}{l} 1 \bmod 3 = 1 \\ 11 \bmod 3 = 2 \\ 111 \bmod 3 = 0 \\ 1111 \bmod 3 = 1 \end{array}\right.$$

$1111 = 370 \cdot 3 + 1$
$-\ 1 = 0 \cdot 3 + 1$
$1110 = 370 \cdot 3$

Pf: Consider the following #'s:

$1, 11, 111, 1111, 11111, \ldots, \underbrace{111\ldots1}_{n+1}$

Consider them all mod n

$\Rightarrow$ there are only n unique mod n values $(0, 1, 2, \ldots, n-1)$

$\Rightarrow$ by PHP, some mod value occurs at least twice.

a & b

$a = q_1 \cdot n + r \qquad b = q_2 \cdot n + r$

$a - b = (q_1 - q_2) \cdot n \to$ a multiple of n.

Generalized PHP: Put n objects into k boxes, then at least one box has at least $\lceil n/k \rceil$ objects.

$\lceil x \rceil$ = "ceiling" of x
= smallest integer at least as large as x.

e.g. 25 objects in 10 boxes,
$\Rightarrow \lceil 25/10 \rceil = \lceil 2.5 \rceil = 3$

Thm: Let $x_1, x_2, \ldots, x_k$ be k non-negative integers where $n = x_1 + x_2 + \cdots + x_k$

and $\overline{x} = \frac{n}{k} = \frac{x_1 + x_2 + \cdots + x_k}{k}$ is their average.

then at least one x_i must be at least as large as $\overline{x}$; i.e., $x_i \geq \overline{x}$.

Pf: Assume for the sake of contradiction that no $x_i \geq \overline{x}$.

Then all $x_i < \overline{x}$.

$$\begin{aligned} x_1 &< \overline{x} \\ x_2 &< \overline{x} \\ &\vdots \\ x_k &< \overline{x} \end{aligned} \Rightarrow x_1 + x_2 + \cdots + x_k < k \cdot \overline{x} \Rightarrow \overline{x} > \frac{x_1 + x_2 + \cdots + x_k}{k} \quad \times$$

Permutations & Combinations

Traveling Salesman Problem: 10 cities to visit, how many ways?

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdots 3 \cdot 2 \cdot 1$$ "10 factorial"

10 choices for first city; 9 for second $= 3{,}628{,}800$

$25! \approx 1.55 \times 10^{25}$

10^{80} atoms in universe

4.12×10^{17} seconds since beginning of universe

$\Rightarrow$ examine 37.6 million possibilities every second since beginning of time

- \# ways to order n objects is $n!$
- each ordering is referred to as a permutation,

\# permutations $= n!$

How many ways to visit 3 out of 10 cities?

$$\underline{10 \cdot 9 \cdot 8} = \frac{10!}{7!} = \frac{10!}{(10-3)!}$$

Def: Permutations: # permutations of n objects is

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Def: r-Permutations: # r-permutations of n objects is

$$_nP_r = P(n, r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdots (n-r+1)}_{r}$$

Example 10 packages to deliver

- how many ways? $10! = 3{,}628{,}800$
- 7 out of 10? $\frac{10!}{(10-7)!} = \frac{10!}{3!} = \frac{10 \cdot 9 \cdot 8 \cdots 4}{7} = 604{,}800$
- 6 on north side - together

  4 on south side - together $\quad 2 \cdot 4! \cdot 6!$

  south or north first

Combinations : an unordered subset of r out of n objects.

$$\#\text{ combinations} = {}_nC_r = C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Pf:

$$P(n,r) = C(n,r)\cdot r!$$

Solve $C(n,r) = \dfrac{P(n,r)}{r!} =$