CS1800
Discrete Structures
Fall 2019

Lecture 19
11/12/19

Last time	Today	Next time
Induction	· Recurrence	· Finish GoF
	· Growth of functions	· Order notation

Merge Sort

3 7 2 5 | 1 6 4 8

2 3 5 7 | 1 4 6 8

1 2 3 4 5 6 7 8

$$T(n) = T(n/2) + T(n/2)$$

$$\overbrace{\hphantom{T(n) = 2 \cdot T(n/2) + n}}^{n}$$

$$\boxed{T(n) = 2 \cdot T(n/2) + n}$$

reality: $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 15n + 6$

but... asymptotically and in terms of order notation, this is equivalent to just

$$T(n) = 2 \cdot T(n/2) + n$$

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$T(n) = 2T(n/2) + n \quad ; \quad T(1) = 1$$

$$T(\boxed{\;}) = \boxed{\;} + 2T(\boxed{\;}/2)$$

pattern

$$\begin{aligned}
T(n) &= n + 2T(n/2) \\
&= n + 2\left[n/2 + 2 \cdot T\left(\frac{n/2}{2}\right)\right] \\
&= n + n + 2^2 \cdot T(n/2^2) \\
&= 2 \cdot n + 2^2 T(n/2^2) \\
&= 2 \cdot n + 2^2 \cdot \left[n/2^2 + 2 \cdot T\left(\frac{n/2^2}{2}\right)\right] \\
&= 2n + n + 2^3 \cdot T(n/2^3) \\
&= 3n + 2^3 \cdot T(n/2^3) \\
&= 4n + 2^4 \cdot T(n/2^4) \\
&\;\vdots \\
&= kn + 2^k \cdot T(n/2^k)
\end{aligned}$$

k times

$\rightarrow$ conjecture for pattern for any k iterations.

Q: For what value of k is $n/2^k = 1$ $\rightarrow$ at this point, recursive procedure terminates with $T(1) = 1$

$\Leftrightarrow \quad n = 2^k$

$\Leftrightarrow \quad k = \log_2 n$

$$T(n) = k \cdot n + 2^k \cdot T(n/2^k)$$

at $k = \log_2 n$

$$= (\log_2 n) \cdot n + 2^{\log_2 n} \cdot T(n/2^{\log_2 n})$$

$$= n \log_2 n + n \cdot T(n/n)$$

$$\log \frac{a}{b} = \log a - \log b$$

$$= n \log_2 n + n \cdot T(1)$$

$$= n \log_2 n + n \cdot 1$$

$$= n \log_2 n + n \quad \checkmark$$

$$T(\square) = \square \log_2 \square + \square$$

$$T(n) = 2 \cdot T(n/2) + n$$

check solution

$$n \log_2 n + n \stackrel{?}{=} 2 \cdot \left[\frac{n}{2} \log_2 \frac{n}{2} + n/2 \right] + n$$

$$= n \log_2 \frac{n}{2} + n + n$$

$$= n \cdot (\log_2 n - \log_2 2) + n + n$$

$$= n \cdot (\log_2 n - 1) + n + n$$

$$= n \log_2 n - \cancel{n} + \cancel{n} + n$$

$$= n \log_2 n + n \quad \checkmark$$

Claim: $\forall k \geq 1$, the iterative pattern

$$T(n) = k \cdot n + 2^k T(n/2^k) \quad \text{holds.}$$

$$T(\square) = \square + 2T(\square/2)$$

Proof (by weak induction):

B.C. $k=1$

$$T(n) = 1 \cdot n + 2^1 T(n/2^1)$$

$$= n + 2 \cdot T(n/2) \checkmark$$

(get back original recurrence)

I.S. Show that <u>if</u> true for $k-1$, then true for k

$$T(n) = (k-1) \cdot n + 2^{k-1} \cdot T(n/2^{k-1})$$

$$= (k-1)n + 2^{k-1}\left[\frac{n}{2^{k-1}} + 2T\left(\frac{n/2^{k-1}}{2}\right)\right]$$

$$= (k-1) \cdot n + n + 2^k T(n/2^k)$$

$$= k \cdot n + 2^k T(n/2^k) \checkmark$$

E.g. $\quad T(n) = 2\cdot T(n/2) + n^2 \quad ; \quad T(1) = 1$

$$\boxed{T(\square) = \square^2 + 2\cdot T(\square/2)}$$

$$T(n) = n^2 + 2T(n/2)$$

$$= n^2 + 2\left[\left(\tfrac{n}{2}\right)^2 + 2\cdot T\left(\tfrac{n/2}{2}\right) \right]$$

$$= n^2 + \frac{n^2}{2} + 2^2 \cdot T(n/2^2)$$

$$= n^2 + \frac{n^2}{2} + 2^2 \cdot \left[\left(\tfrac{n}{2^2}\right)^2 + 2\cdot T\left(\tfrac{n/2^2}{2}\right) \right]$$

$$= n^2 + \frac{n^2}{2} + \frac{n^2}{2^2} + 2^3 \cdot T(n/2^3)$$

$$= n^2 + \frac{n^2}{2} + \frac{n^2}{2^2} + 2^3 \left[(n/2^3)^2 + 2\cdot T\left(\tfrac{n/2^3}{2}\right) \right]$$

$$= n^2 + \frac{n^2}{2} + \frac{n^2}{2^2} + \frac{n^2}{2^3} + 2^4 \cdot T(n/2^4)$$

$$= n^2\left(1 + 1/2 + 1/2^2 + 1/2^3 + 1/2^4\right) + 2^5 \cdot T(n/2^5)$$

$$\vdots$$

k times $\quad = n^2\left(1 + 1/2 + 1/2^2 + 1/2^3 + \dots + 1/2^{k-1}\right) + 2^k T(n/2^k)$

$$= n^2 \sum_{i=0}^{k-1} 1/2^i + 2^k T(n/2^k)$$

$$\frac{n}{2^k} = 1 \iff k = \log_2 n$$

$$T(n) = n^2 \sum_{i=0}^{k-1} \frac{1}{2^i} + 2^k\, T(n/2^k)$$

$$\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$$

$k = \log_2 n$

$$= n^2 \sum_{i=0}^{\log_2 n - 1} \frac{1}{2^i} + 2^{\log_2 n}\, T(n/2^{\log_2 n})$$

$$= n^2 \sum_{i=0}^{\log_2 n - 1} (1/2)^i + n \cdot T(1)$$

$$= n^2 \cdot \sum_{i=0}^{\log_2 n - 1} (1/2)^i + n$$

$$= n^2 \frac{1-(1/2)^{\log_2 n}}{1-1/2} + n$$

$$= n^2 \cdot \frac{1 - \frac{1}{2^{\log_2 n}}}{1/2} + n$$

$$= n^2 \frac{1-1/n}{1/2} + n = n^2(2 - 2/n) + n$$

$$= 2n^2 - 2n + n = \boxed{2n^2 - n}$$

check $T(n) = 2T(n/2) + n^2$

$$2n^2 - n \overset{?}{=} 2\left[2\cdot(n/2)^2 - (n/2)\right] + n^2$$

$$= n^2 - n + n^2$$

$$= 2n^2 - n \quad ✓$$

Claim: $\forall k \geq 1$, the iterative pattern

$$T(n) = n^2 \cdot \sum_{i=0}^{k-1} \frac{1}{2^i} + 2^k \, T(n/2^k) \text{ holds.}$$

$$T(B) = B^2 + 2 \cdot T(B/2)$$

Pf by induction: B.C. $k=1$

$$T(n) = n^2 \cdot \sum_{i=0}^{1-1} \frac{1}{2^i} + 2^1 \cdot T(n/2^1)$$

$$= n^2 \cdot \frac{1}{2^0} + 2\,T(n/2)$$

$$= n^2 + 2\,T(n/2) \checkmark$$

I.S. Show that if true for $k-1$, then true for k

$$T(n) = n^2 \sum_{i=0}^{k-2} 1/2^i + 2^{k-1} \, T\left(\frac{n}{2^{k-1}}\right)$$

$$= n^2 \sum_{i=0}^{k-2} \frac{1}{2^i} + 2^{k-1}\left(\left(\frac{n}{2^{k-1}}\right)^2 + 2\,T\left(\frac{n/2^{k-1}}{2}\right)\right)$$

$$= n^2 \sum_{i=0}^{k-2} \frac{1}{2^i} + \frac{n^2}{2^{k-1}} + 2^k \, T(n/2^k)$$

$$= n^2 \sum_{i=0}^{k-1} \frac{1}{2^i} + 2^k \, T(n/2^k) \checkmark$$

$$T(n) = 4T(n/2) + n$$

$$T(\square) = \square + 4T(\square/2)$$

$$T(n) = n + 4T(n/2)$$

$$= n + 4\left(\frac{n}{2} + 4 \cdot T(n/2^2)\right)$$

$$= n + \frac{4}{2}n + 4^2 T(n/2^2)$$

$$= n + \frac{4}{2}n + 4^2\left[\frac{n}{2^2} + 4T\left(\frac{n/2^2}{2}\right)\right]$$

$$= n + \frac{4}{2}n + \frac{4^2}{2^2}n + 4^3 T(n/2^3)$$

$$= n + \frac{4}{2}n + \left(\frac{4}{2}\right)^2 n + \left(\frac{4}{2}\right)^3 n + 4^4 T(n/2^4)$$

$$\vdots$$

$$= n + \left(\frac{4}{2}\right)n + \left(\frac{4}{2}\right)^2 n + \cdots + \left(\frac{4}{2}\right)^{k-1} n + 4^k T(n/2^k)$$

$$= n\left(1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1}\right) + 4^k T(n/2^k)$$

$$= n \cdot \sum_{i=0}^{k-1} 2^i + 4^k T(n/2^k)$$

$$\frac{n}{2^k} = 1 \iff k = \log_2 n$$

$$T(n) = n \cdot \sum_{i=0}^{k-1} 2^i + 4^k\, T\left(n/2^k\right)$$

$k = \log_2 n$

$$= n \cdot \sum_{i=0}^{\log_2 n - 1} 2^i + 4^{\log_2 n}\, T\left(n/2^{\log_2 n}\right)$$

$$= n \cdot \frac{1 - 2^{\log_2 n}}{1-2} + n^{\log_2 4} \cdot T(1)$$

$$= n \cdot \frac{1-n}{-1} + n^2 \cdot 1$$

$$= n(n-1) + n^2$$

$$= n^2 - n + n^2$$

$$= \boxed{2n^2 - n}$$

Facts

(1) $\sum_{i=0}^{k} r^i = \dfrac{1-r^{k+1}}{1-r}$

(2) $a^{\log_b c} = c^{\log_b a}$

- to show correct, take $\log_b$ of both sides.

To finish proof, prove the pattern correct by induction.