CS1800
Discrete Structures
Fall 2019

Lecture 18
11/8/19

Last time

- Finished series
  - telescoping
- Start induction
  - weak
  - strong

Today

- More induction

Next time

- Finish induction
- Recurrences

Q: How many bit strings of length n have <u>no</u> consecutive 1s?

n=1   0, 1   2

n=2   00, 01, 10, ~~11~~   3

n=3   000, 001, 010, ~~011~~, 100, 101, ~~110~~, ~~111~~   5

n=4   0000, 0001, 0010, ~~0011~~, 0100, 0101, ~~0110~~, ~~0111~~   1000, 1001, 1010, ~~1011~~, ~~1100~~, ~~1101~~, ~~1110~~, ~~1111~~   8

F_1	F_2	F_3	F_4	F_5	F_6	F_7
1	1	2	3	5	8	13

<u>Conjecture</u>: # such strings $S(n) = F_{n+2}$

Pf: Ind step: Show if true $\forall k < n$, then true at n;

i.e., $S(k) = F_{k+2}$ $\forall k < n$, then show that this implies that $S(n) = F_{n+2}$.

- Every bit string of length n w/o consecutive 1s <u>either</u> begins w/ a <u>0</u> or a <u>1</u>
- If begins w/ 0 $0\underbrace{xxxx\cdots x}_{n-1}$ ↳ must not contain consec. 1s

  how many? $S(n-1) = F_{n+1}$ by ind. hyp.

- If begin w/ 1, $10\underbrace{xxxxx\cdots x}_{n-2}$ (must be 0) ↳ must not contain consec. 1s

  next bit must be 0.

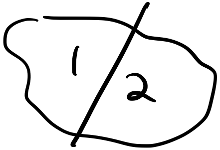
  remainder: no consec. 1s $\Rightarrow S(n-2) = F_n$

- $S(n) = F_{n+1} + F_n = F_{n+2}$ ✓
- Need 2 bases cases, but already proven for n=1, n=2

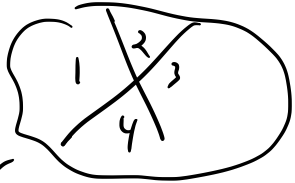
E.g. · n lines in the plane w/ no 2 lines parallel and no 3 lines concurrent.

· how many regions does this divide the plane into?

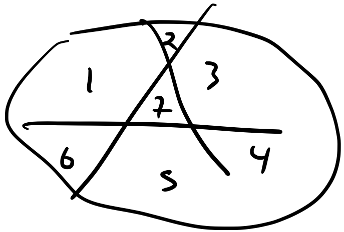
$n=1$:



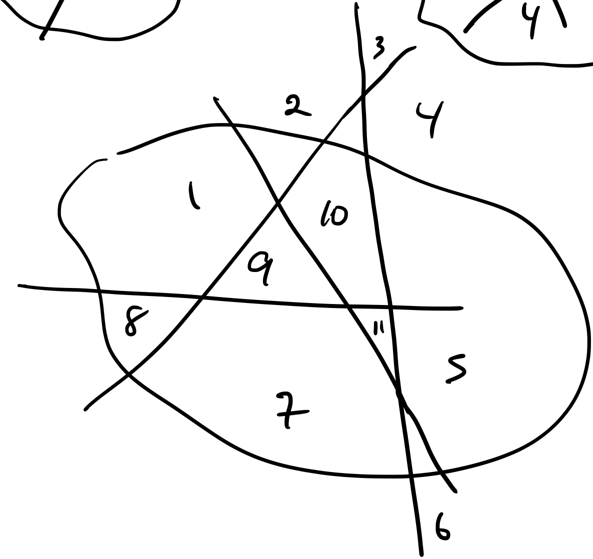
$n=2$:



$n=3$:



$n=4$:



n:	1	2	3	4	...
#:	2	4	7	11	...

$$2 \quad 3 \quad 4$$

$$1 \quad 1$$

- try quadratic sequence, and then <u>prove</u> w/ induction
- $r_n = a \cdot n^2 + b \cdot n + c$ for some a, b, c

$$r_1 = a \cdot 1^2 + b \cdot 1 + c = 2 \Rightarrow a+b+c=2$$
$$r_2 = a \cdot 2^2 + b \cdot 2 + c = 4 \Rightarrow 4a+2b+c=4$$
$$r_3 = a \cdot 3^2 + b \cdot 3 + c = 7 \Rightarrow 9a+3b+c=7$$

$3a+b=2$

$5a+b=3$

$2a=1 \Rightarrow a=1/2$

$b=1/2$

$c=1$

$$\Rightarrow r_n = 1/2 \cdot n^2 + 1/2\, n + 1 = \frac{n^2+n+2}{2}$$

- check for $n=4$ $\quad r_4 = \frac{4^2+4+2}{2} = \frac{16+4+2}{2} = \frac{22}{2} = 11$ ✓

- But what about $n = 5, 6, 7, \ldots$?

$\Rightarrow$ prove by <u>induction</u>.

Claim: $r_n = \frac{n^2+n+2}{2}$

Pf: Induction

Base case: $n=1$ $\qquad r_1 = \frac{1^2+1+2}{2} = \frac{4}{2} = 2$ ✓

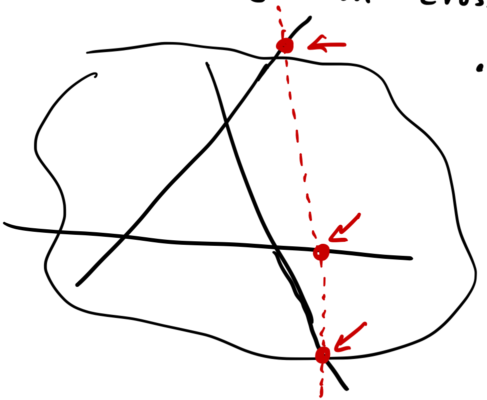
Ind. step: show that if true for $n=k$, $\rightarrow r_k = \frac{k^2+k+2}{2}$

then true at $n=k+1$. $\rightarrow r_{k+1} = \frac{(k+1)^2+(k+1)+2}{2}$

$$= \frac{k^2+2k+1+k+1+2}{2}$$

$$= \frac{k^2+3k+4}{2}$$

- For any $k+1$ lines, remove 1 of them.
- what remains is k lines and $\frac{k^2+k+2}{2}$ regions, by I.H.
- the $k+1^{st}$ line must cross all k lines at unique points



- the $k+1^{st}$ line crosses k lines, dividing $k+1$ regions in half, creating $k+1$ additional regions.

- $r_{k+1} = r_k + (k+1)$

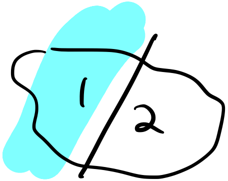
$$= \frac{k^2+k+2}{2} + (k+1)$$

$$= \frac{k^2+k+2+2k+2}{2} = \frac{k^2+3k+4}{2}$$ ✓

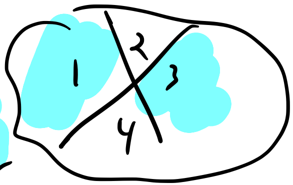
E.g. Same setup: n lines in plane, no 2 parallel, no 3 concurrent.

Claim: Can always color the regions black & white s.t. no 2 adjacent regions have same color.

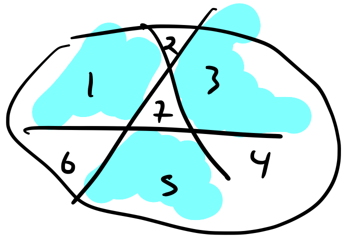
$n=1$:



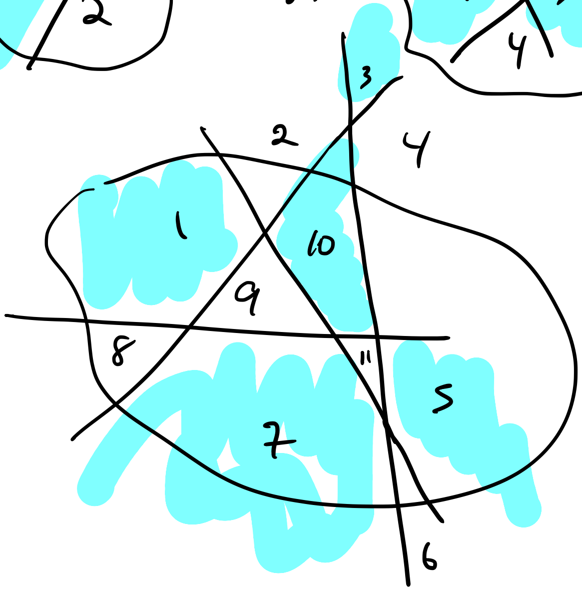
$n=2$:



$n=3$:



$n=4$:



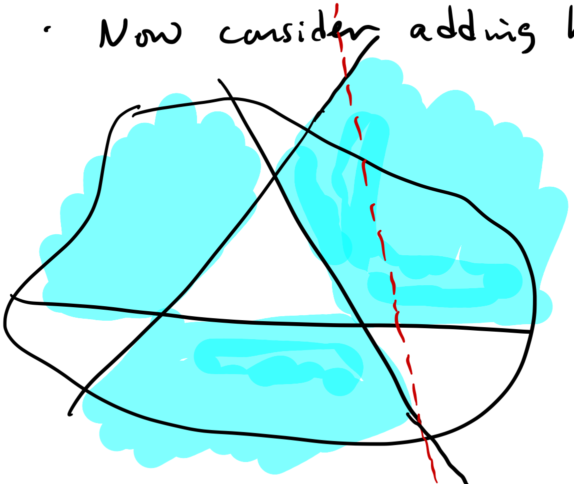
- Proof by induction

- B.C. $n=1$ ✓

- I.S. Show if true for $n=k$, ← can color the $\frac{k^2+k+2}{2}$ regions using just B & W
  then true for $n=k+1$ ← can color the $\frac{k^2+3k+4}{2}$ regions using just B & W.

- For any $k+1$ lines, remove 1. Left w/ k lines and can color the regions w/ just 2 colors by I.H.

- Now consider adding $k+1^{st}$ line back in.



  - every region to the left of red (new $k+1^{st}$) line is colored OK w.r.t. each other. Same is true to the right. Problem is across new line.
  - solution: swap all colors on one side only.
  - preserves correctness on that side
  - makes correct across line.