CS1800
Discrete Structures
Fall 2019

Lecture 17
11/5/19

Last time

- Series
  - arithmetic
  - geometric
  - harmonic

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- Fibonacci math trick

Today

- Series
  - telescoping

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- Induction
  - weak
  - strong

Next time

- More induction

Telescoping Series

$$\sum_{k=0}^{n} (a_k - a_{k+1}) = (a_0 - a_1) + (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_{n+1})$$

$$= a_0 - \cancel{a_1} + \cancel{a_1} - \cancel{a_2} + \cancel{a_2} - a_3 \cdots$$

$$= a_0 - a_{n+1}$$

$$S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots$$

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots + \frac{1}{(n-1)\cdot n} = \sum_{k=1}^{n-1} \frac{1}{k(k+1)}$$

$a_k = 1/k$

Claim: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ $\qquad \frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$

$$\Rightarrow \sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} \cdots = 1 - \frac{1}{(n-1)+1} = 1 - 1/n$$

Induction

$S(n)$: $\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2} \quad \forall n \geq 1$

$S(n)$: Can always make postage for n cents using just 4 & 5 ¢ stamps $\forall n \geq 12$

Weak induction

Base case, B.C., show that $S(1)$

Inductive step, I.S., show that $S(k) \Rightarrow S(k+1)$

$$S(1) \Rightarrow S(2) \Rightarrow S(3) \Rightarrow S(4) \Rightarrow \cdots$$

B.C. I.S. I.S I.S.

Strong induction

B.C. $S(1), S(2), S(3), \ldots, S(b)$

I.S. $S(1), S(2), \ldots S(k-1) \Rightarrow S(k)$

$S(1)$

$\Rightarrow S(1), S(2)$

$\Rightarrow S(1), S(2), S(3)$

$\vdots$

Weak Induction

E.g. $S(n): \sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2} \quad \forall n \geq 1$

Base Case: Show true for $n=1$

$$\sum_{i=1}^{1} i = 1 \quad \equiv \quad \frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1 \quad \checkmark$$

$S(1)$ is true.

Inductive Step: Show that if true at $n=k$, $\rightarrow \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

then true at $n = k+1$ $\rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$(S(k) \Rightarrow S(k+1))$.

to prove

inductive hypothesis (assumption)

$$\sum_{i=1}^{k+1} i = \underbrace{1+2+3+\cdots+k}_{\sum_{i=1}^{k} i} + (k+1)$$

$$= \sum_{i=1}^{k} i + (k+1)$$

inductive hypothesis

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \quad \checkmark$$

E.g. $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} = 1 - \frac{1}{n+1} \quad \forall n \geq 1$

Base case: $n=1$ show $\displaystyle\sum_{i=1}^{1} \frac{1}{i(i+1)} \stackrel{?}{=} \frac{1}{1+1}$

$$\downarrow$$

$$\frac{1}{1\cdot(1+1)} \stackrel{?}{=} \frac{1}{1+1} \quad \checkmark$$

Inductive step: Show if true for $n=k$, $\rightarrow \displaystyle\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} \leftarrow (*)$

then true at $n=k+1$ $\rightarrow \displaystyle\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2} \leftarrow *$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$$

ind. hyp.

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} \quad \checkmark$$

E.g. $\displaystyle\sum_{i=1}^{n} (2i-1) = n^2 \quad \forall n \geq 1$

$n=3 \quad 1+3+5 = 3^2$

$n=4 \quad 1+3+5+7 = 4^2$

$n=5 \quad 1+3+5+7+9 = 5^3$

Base case: $n=1$

$$\sum_{i=1}^{1} (2i-1) \stackrel{?}{=} 1^2$$

$$2\cdot 1-1 = 1 \qquad 1 \qquad \checkmark$$

Inductive step: Show if true for $n=k$, $\leftarrow$ $\displaystyle\sum_{i=1}^{k} (2i-1) = k^2$

then must be true for $n=k+1$ $\leftarrow$ $\displaystyle\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$

$$\sum_{i=1}^{k+1} (2\cdot i-1) = \sum_{i=1}^{k} (2\cdot i-1) + (2\cdot(k+1)-1)$$

$$= k^2 + (2k+2-1)$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2 \quad \checkmark$$

Postage Stamp Problem

Claim: Can always make postage for n cents using just 4 & 5 ¢ stamps. $\forall n \geq 12$

Try weak induction...

B.C. $n = 12$ $\quad 3 \times 4¢ = 12¢$ ✓

I.S. Show if true at $n = k$, → I.H.: Assume can make change for k cents using just 4 & 5 ¢ stamps

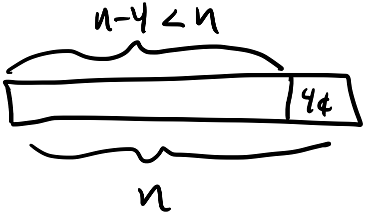
then true at $n = k+1$ → Prove that can make change for $k+1$ cents using just 4 & 5 ¢ stamps

⇒ in other words, prove true at $n = k+1$ relying only on true at $n = k$ (and algebra).

Proof fails: at $n = k+1$, use either 4 or 5 ¢ stamp, but then have an $n = k-3$ or $n = k-4$ problem (so ind. hyp. does not apply)

Strong Induction

Ind. Step:
- Show that if true for all $12 \le k < n$, then true at n.



- Proof:
  - Use 4¢ stamp. Then must make postage for $n-4$¢ using just 4 & 5¢ stamps.
  - Since $n-4 < n$, this can always be done using 4 & 5¢ stamps by ind. hyp. ✓

Base cases:

$n=12$	3x 4¢
$n=13$	2x4¢ 1x5¢
$n=14$	1x4¢ 2x5¢
$n=15$	3x 5¢

✓