CS1800
Discrete Structures
Fall 2019

Lecture 16
11/1/19

Last time

- Finished motivation
  - sorting
- Sequences
  - arithmetic
  - geometric
  - quadratic

Today

- Series
  - arithmetic
  - geometric
    + infinite

optional

  - harmonic
  - telescoping
  - math trick
    - Fibonacci Representations

Next time

- Skip lists
- Induction

Series: Sum of elements from a sequence

E.g. Arithmetic sequence: $1, 2, 3, 4, \_\_ n$

Arithmetic Series: $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$

E.g. Geometric series: $3 + 3/2 + 3/4 + 3/8 + 3/16 + \cdots = ?$

Arithmetic Series: Gauss's Trick

$$
\begin{aligned}
S &= 1 + 2 + 3 + \cdots + 100 \\
+\quad S &= 100 + 99 + 98 + \cdots + 1 \\
\hline
2 \cdot S &= \underbrace{101 + 101 + 101 + \cdots 101}_{100}
\end{aligned}
$$

$$= 100 \cdot 101 = 10100$$

$$S = 10100 \div 2 = 5050$$

Examples

①

$$S = 1 + 2 + 3 + 4 + \cdots + n$$

$$+ \quad S = n + (n-1) + (n-2) \cdots + 1$$

$$2S = \underbrace{(n+1) + (n+1) + \cdots (n+1)}_{n}$$

$$= n\cdot(n+1)$$

$$\Rightarrow 2S = n(n+1) \Rightarrow S = \frac{n\cdot(n+1)}{2}$$

$$\longrightarrow \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

②

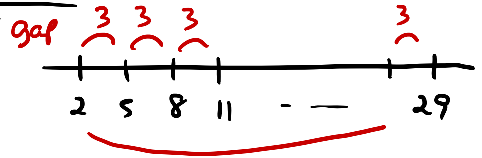
$$S = 2 + 5 + 8 + 11 + 14 + \cdots + 29$$

$$S = 29 + 26 + 23 + \cdots + 2$$

$$2S = \underbrace{31 + 31 + 31 + \cdots + 31}_{10}$$

$$= 31\cdot 10$$

$$\Rightarrow S = \frac{31\cdot 10}{2}$$



span = 29 − 2 = 27

$$\#\text{gaps} = \frac{\text{span}}{\text{gap size}} = \frac{27}{3} = 9$$

$$\#\text{numbers} = \#\text{gaps} + 1$$

$$= 10$$

3

$$S = -11 - 7 - 3 + 1 + 5 + 9 + \cdots + 17$$

$$S = 17 + 13 + 9 + \cdots \quad -11$$

$$2S = \underbrace{6 + 6 + 6 + \cdots + 6}$$

$$= \frac{(17 - (-11))}{4} + 1 = \frac{28}{4} + 1 = 7 + 1 = 8$$

$$2S = 6 \cdot 8$$

$$S = \frac{6 \cdot 8}{2} = 24$$

Geometric Series: $3 + 3/2 + 3/4 + \cdots + 3/32$

$$S = 3 + \cancel{3/2} + \cancel{3/4} + \cancel{\cdots} + \cancel{3/32}$$

$$-\quad 1/2 \cdot S = \cancel{3/2} + \cancel{3/4} + \cancel{3/8} + \cdots \cancel{} + 3/64$$

$$S - 1/2 \cdot S = 3 - 3/64$$

$$1/2\, S = 3 - 3/64$$

$$S = 6 - 3/32$$

Constant ratio r ; Start at 1

$r = 1/2$

$$S = 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots \cancel{r^n}$$

$$1 + 1/2 + 1/4 + 1/8 + \cdots + 1/2^n$$

$$- r \cdot S = \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots \cancel{r^n} + r^{n+1}$$

$$S - r \cdot S = 1 - r^{n+1}$$

$$\downarrow$$

$$(1-r) \cdot S = 1 - r^{n+1}$$

$$S = \frac{1 - r^{n+1}}{1-r}$$

$$S = \sum_{i=0}^{n} r^n = \frac{1 - r^{n+1}}{1-r}$$

$r = 2$

$$S = 1 + 2 + 4 + 8 + \cdots 2^n$$

$$S = \frac{1 - 2^{n+1}}{1-2} = \frac{1 - 2^{n+1}}{-1} = 2^{n+1} - 1$$

Infinite geometric series, e.g.

$$S = 1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k \qquad |r|<1$$

e.g. $1 + 1/2 + 1/4 + 1/8 + \cdots$

not precise

$$\begin{aligned} S &= 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots \\ - r\cdot S &= \quad \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots \end{aligned}$$

$$S - r\cdot S = 1$$

$$(1-r)\cdot S = 1$$

$$S = \frac{1}{1-r}$$

$$\frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

precisely...

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k$$

$$= \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r}$$

$$= \frac{1}{1-r}$$

because $\lim_{n\to\infty} r^{n+1} \to 0 \quad |r|<1$

③ Harmonic Series

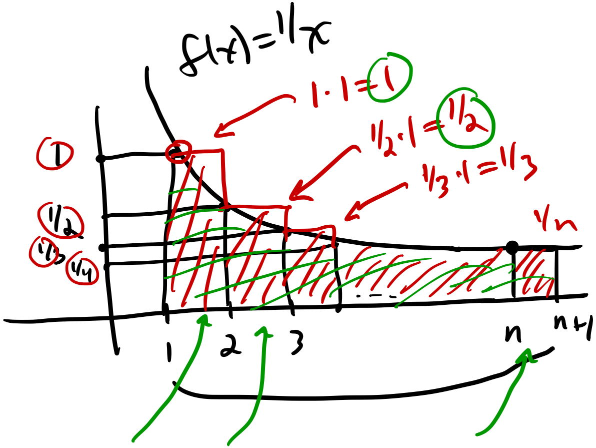
$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

$$= \ln(n) + \sim \text{constant}$$

$\hookrightarrow \gamma \sim 0.577$

e.g. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1000} \sim \ln(1000)$

why



(red) – area of rectangles

(green) – area under curve

→ about the same

(red) $\approx \int_1^{n+1} \frac{1}{x}\,dx$

$$= \ln x \Big|_1^{n+1} = \ln(n+1) - \ln 1$$

$$= \ln(n+1) \checkmark$$

Fibonacci Numbers

Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

$2 \quad 1.5 \quad 1.66 \quad 1.6 \quad 1.625 \quad 1.615 \quad 1.619 \quad 1.617$

$$F_n = \begin{cases} 1 & n = 1 \text{ or } 2 \\ F_{n-1} + F_{n-2} & n > 2 \end{cases}$$

Ratio converges to Golden Ratio

$$\phi = \frac{1+\sqrt{5}}{2} \approx \underline{1.618}$$

Representing numbers in Fibonacci

	1			1			1			1	
			1			1			1		1
	144	89	55	34	21	13	8	5	3	2	1
47 =				1	0	1	0	0	0	0	0
26 =					1	0	0	1	0	0	0
40 =				1	0	0	0	1	0	0	1

Coincidence: # km/mi ≈ 1.609

47 miles → km

$$47 = 34 + 13 \rightarrow 1.6 \times (47) = \underbrace{1.6 \cdot 34}_{55} + \underbrace{1.6 \times 13}_{21} = 76$$