CS1800

Discrete Structures

Fall 2019

Lecture 10

10/8/19

Last time

- Finish counting

Today

- Start probability

Next time

- Continue probability

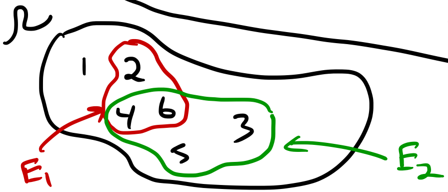
Quiz

- translate English statements to logic statements
- negate logic statements
- Tarski world like problem
  - predicates

Probability

- Random experiment
- generate outcomes $\omega \in \Omega$
- sample space: set of all possible outcomes Ω
- Event: subset of sample space
- $p : \Omega \to \mathbb{R}$ probability measure
  - $0 \leq p(\omega) \leq 1 \quad \forall \omega \in \Omega$
  - $\sum_{\omega \in \Omega} p(\omega) = 1$

we will assume for now that $p(\omega) = \frac{1}{|\Omega|}$ for $\forall \omega \in \Omega$



Example

- roll a fair six-sided die
- roll a 5
- $\{1, 2, 3, 4, 5, 6\} = \Omega$
- $E_1 = \text{"even"} = \{2, 4, 6\}$

  $E_2 = \text{"}\geq 3\text{"} = \{3, 4, 5, 6\}$
- $p(1) = p(2) = p(3) = \cdots = p(6) = 1/6$

$$P(E) = \sum_{\omega \in E} p(\omega)$$

e.g. $P(E_1) = p(2) + p(4) + p(6) = 1/6 + 1/6 + 1/6$
$= 1/2$

If $p(\omega) = \frac{1}{|\Omega|} \ \forall \omega \in \Omega$

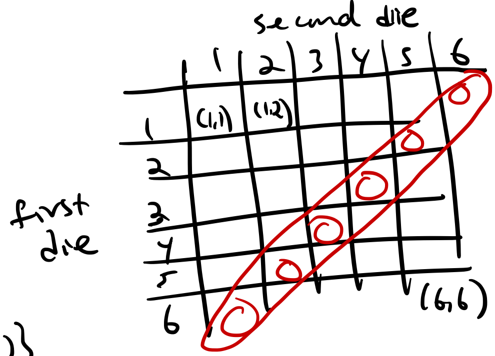
$$P(E) = |E| / |\Omega| \qquad P(E_1) = \frac{|E|}{|\Omega|} = \frac{3}{6}$$

Examples

① Roll one fair die

$\Omega = \{1,2,3,4,5,6\}$

② Roll two fair die



$\Omega = \{(1,1),(1,2),(1,3),\ldots(2,1),(2,2),\ldots(6,6)\}$

$= \{1,2,3\ldots6\} \times \{1,2,\ldots,6\}$

$E_1 = \text{total is } 7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{6}{36} = 1/6$

$E_2 = \text{total is greater than } 8$

$= 9 \text{ or } 10 \text{ or } 11 \text{ or } 12$

$|E_2| = \underset{12}{1} + \underset{11}{2} + \underset{10}{3} + \underset{9}{4}$

$= 10$

$P(E_2) = \frac{|E_2|}{|\Omega|} = \frac{10}{36} = \frac{5}{18}$

Cards:

Standard deck of cards

- 4 suits Hearts, Diamonds, Clubs, Spades
- w/i each suit 2, 3, 4, ..., 10, J, Q, K, A
- 13 cards per suit
- $\Rightarrow 13 \cdot 4 = 52$ cards total

- Rand. Exp. - draw one card from deck

- Sample space:

$$\Omega = \{2H, 3H, \ldots AH, 2D, 3D, \ldots, \ldots, AS\}$$

$$|\Omega| = 52$$

- E_1 = "face" card (J, Q, K, A)

$$|E_1| = \underset{\#\text{ suits}}{4} \cdot \underset{J,Q,K,A}{4} = 16$$

$$P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{16}{52} = \frac{8}{26} = \frac{4}{13}$$

- E_2 = card is between 2 and 10 (number)

$$|E_2| = 9 \cdot 4 = 36$$

$$P(E_2) = \frac{|E_2|}{|\Omega|} = \frac{36}{52} = 9/13$$

Urn Problems :

15 red cubes
10 blue cubes

- Rand. Exp. : draw one cube from urn

$$\Omega = \{ R1, R2, \ldots, R15, B1, B2, \ldots B10 \} \qquad |\Omega| = 25$$

- $E_1 = \text{red}$ $\qquad E_1 = \{ R_1, R_2, \ldots R15 \}$

$$|E_1| = 15 \qquad P(E_1) = \frac{15}{25} = 3/5$$

- Rand. Exp. : draw 3 cubes at once (sampling w/o replacement)

$$\Omega = \{ \{R_1, R_2, R_3\}, \{R_1 R_2 R_4\}, \ldots \qquad \{B_8, B_9, B_{10}\} \}$$

$$|\Omega| = \binom{25}{3} = 2300 \qquad P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{\binom{15}{3}}{\binom{25}{3}} = \frac{455}{2300} \approx 19.8\%$$

- $E_1 = \text{all red}$ $\qquad |E_1| = \binom{15}{3} = 455$

15 red cubes

10 blue cubes

- E_2 = 2 red, 1 blue

$$|E_2| = \binom{15}{2} \cdot \binom{10}{1} = 1050$$

$$P(E_2) = \frac{\binom{15}{2}\binom{10}{1}}{\binom{25}{3}} = \frac{1050}{2300} \approx 45.7\%$$

- Sampling w/ replacement　　　25 cubes, 15 red, 10 blue

↳ Draw 3 cubes, one at a time, put back between draws

$\Omega = \{ (R1, R1, R1), (R1, R1, R2), \ldots \quad (B10, B10, B10) \}$

$|\Omega| = 25^3$

- E_1 = all red　　$|E_1| = 15^3$

$$P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{15^3}{25^3} = \frac{27}{125} \approx 21.6\%$$

- E_2 = 2 red & 1 blue　　$|E_2| = 3 \cdot 10 \cdot 15^2$

choose which position for blue

pick reds

pick blue cube

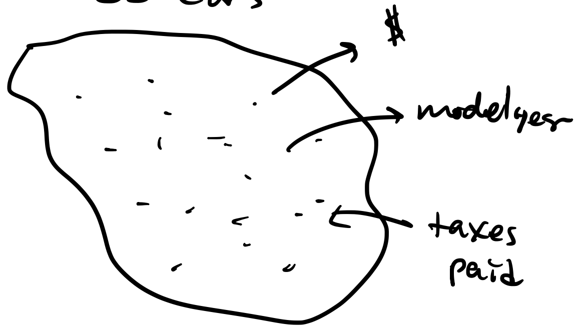
$$P(E_2) = \frac{|E_2|}{|\Omega|} = \frac{3 \cdot 10 \cdot 15^2}{25^3} \approx 43.2\%$$

Expectation

- Random variable

$X : \Omega \to \mathbb{R}$

Ω = Cars



$$E[X] = \sum_{x} x \cdot \Pr[X = x]$$

expected or "average" value

$$E(X) = \sum_{\omega \in \Omega} X(\omega) \cdot p(\omega)$$

1/2 cars $27,000

1/3 cars $10,000

1/6 cars $15,000

$$E[X] = \sum_{x} x \cdot \Pr[X = x] = \tfrac{1}{2} \cdot 2700 + \tfrac{1}{3} \cdot 10{,}000 + \tfrac{1}{6} \cdot 1500 = \$19{,}333.33$$

Exaple

- Pay \$6 to play game
- Roll two fair 6-sided die
- Pay you sum of die faces

  except if doubles, then 0
- X = winnings (profit)
- $E[X] = \sum_{x} x \cdot Pr\{X=x\}$

$x=-6 \quad Pr\{X=-6\} = 6/36 = 1/6$

$x=-3 \quad Pr\{X=-3\} = 2/36 = 1/18$

$\vdots$

$x=+5 \quad Pr\{X=5\} = 2/36 = 1/18$

$$E[X] = \sum_{x} x \cdot Pr\{X=x\}$$

$$= (-6)\cdot 1/6 + (-3)\cdot 1/18 + \cdots + 5(1/18) = -0.16\overline{6}$$

lose 16 ¢ per play on average

die 2

die 1	1	2	3	4	5	6
1	-6	-3	-2	-1	0	1
2	-3	-6	-1	0	1	2
3	-2	-1	-6	1	2	3
4	-1	0	1	-6	3	4
5	0	1	2	3	-6	5
6	1	2	3	4	5	-6

- $E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot p(\omega)$ $\quad$ 1/36

$$= \frac{1}{36} \sum_{\omega \in \Omega} X(\omega)$$

$$= \frac{1}{36} \left(\text{"sum whole table"} \right)$$

$$= -0.16\overline{6}$$