Problem 6 [Hard]**: Divisibility and indexed sets**

Recall that $n \in \mathbb{Z}$ is divisible by k if there exists $b \in \mathbb{Z}$ such that $n = bk$. When counting multiples of k in a given range, it is often easier (and safer) to index the set. For example, the set of integers divisible by $k = 7$ between 1 and 50 is
$A = \{7, 14, 21..., 49\} = \{7i | i \in \mathbb{Z}, 1 \leq i \leq 7\}$, the last expression being the set indexed by i from 1 to 7. Once a set is indexed *starting at 1*, it is easy to count: since indices go from 1 to 7, set A has 7 elements.

$\{7, 14, ..., 49\} \leftarrow i \in \{1, 2, 3, 4, 5, 6, 7\}$

map(i)=7i

Another example: let's say we want to count the set of integers divisible by 13 between 100 and 300. We index the set as

1,2,3,4...,16

$B = \{104, 117, 130..., 299\} = \{13j + 91 | j \in \mathbb{Z}, 1 \leq j \leq 16\}$. Verify the first `13*1+91=104` and the last `13*16+91=299`. Since indices go from 1 to 16, we have $|B| = 16$.

For each of the following questions, explain your reasoning for full credit. ?

A B C

i. How many positive integers from 1 to 500 are divisible by 2 and 3 but not 5?

ii. How many positive integers from 1 to 500 are divisible by 2 or 3 or 11? $\rightarrow |A \cup B \cup C|$

A B C

iii. What is the least number of distinct integers we can choose between 1 and 500, that guarantees that at least one of them is divisible by 7?

Permutations / Combinations.

① Permute n elements/obj on n spots

$$\underset{n}{\underline{\quad\quad}}\ \underset{n-1}{\underline{\quad\quad}}\ \underset{n-2}{\underline{\quad\quad}}\ \underline{\quad}\ \underset{2}{\underline{\quad\quad}}\ \underset{1}{\underline{\quad\quad}} \Rightarrow n! \text{ possib.}$$

② Fix $k \le n$, do the same (all permutations)

$$\underbrace{\boxed{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}}_{k}\ \underbrace{\boxed{\underline{\quad}\ \underline{\quad}\ \underline{\quad}}}_{n-k}$$

$\boxed{\text{Generative Process}}$ $\Rightarrow$ usually product rule

1. Choose a subset of $k \Rightarrow \binom{n}{k}$
2. Permute these k on first k spots $\Rightarrow k!$
3. Permute the other $n-k$ items on last $n-k$ spots $\Rightarrow (n-k)!$

This process generates all permutations exactly once

$$\to \binom{n}{k} \cdot k! \cdot (n-k)! = n! \Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

product rule

$\binom{n}{k} \stackrel{def}{=}$ # ways to pick k items out of order from n

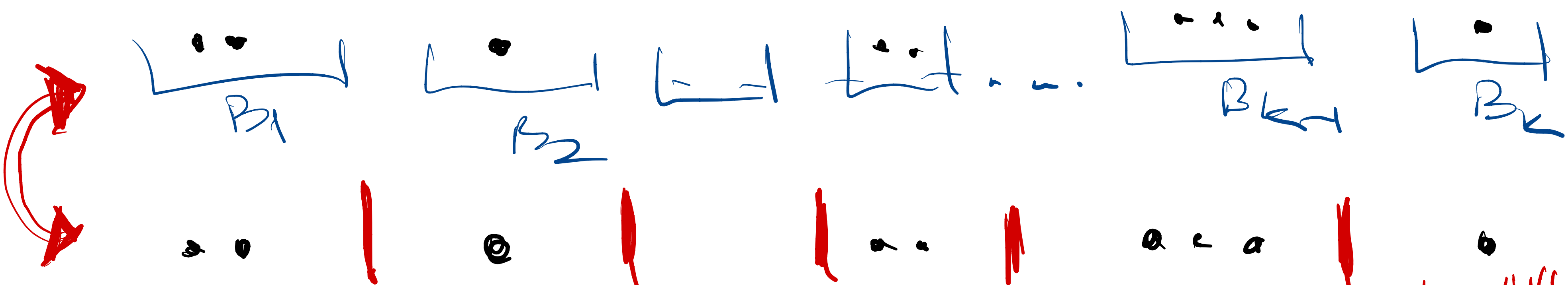
= # subsets of size k of any subset size n.

③ Balls into bins

n balls (identical, not distinguished)

k bins $B_1, B_2, \dots, B_k$

How many ways to distribute the balls (counts) into these k bins?



$k-1$ separators "|"

Balls into bins $\Longleftrightarrow$ n ($\bullet$) with $k-1$ (|)

particular n choose k $\Longleftrightarrow$ $n+k-1$ total symbols, choose $k-1$ spots for "|"

$$\Longleftrightarrow \binom{n+k-1}{k-1}$$

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2 main counting strategies (3 including Balls-in-Bins)

① Product Rule (generative process)

- break generation of outcomes into individual independent choices.
- count each choice options
- product
- take out some extra stuff if neccessary
  - invalid overall choices
  - double counting

② Sum Rule : break problem into disjoint cases

ex count all licence plates 6-digits that end with odd digit.

5 cases

count$_1$ ← _ _ _ _ _ 1

count$_2$ ← _ _ _ _ _ 3

count$_3$ ← _ _ _ _ _ 5

count$_4$ ← _ _ _ _ _ 7

count$_5$ ← _ _ _ _ _ 9

disjoint
~~no~~ no overlap

$\sum_i$ counts = answer.

subtract unwanted outcomes

Product rule

$\underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{5}$ / odd

$\Rightarrow 10^5 \cdot 5.$

CS1800
Discrete Structures
Fall 2017

Lecture 19
10/19/17

$x=7, y=2z^{-2}, n=5$ | $x=5$ $y=1.3$ $n=10$

$(7+2z^{-2})^5$ | $(5+1.3)^{10}$

Last time

- ~~Finish Perm. & Comb.~~
  - ~~examples~~
  - ~~balls-in-bins~~

Today

$\boxed{\text{Binomial Theorem}}$ → Binomial Distribution

Binomial Expansion

$x, y \in \mathbb{R}$

Next time

- Counting Problem Examples

$$(x+y)^n = \underset{\text{1st pair } (\ )}{(x+y)}\underset{\text{2nd } (\ ) \text{ pair}}{(x+y)}\underset{\text{3rd}}{(x+y)} \cdots \underset{(n-1)^{th}}{(x+y)}\underset{n^{th}}{(x+y)}$$

open ( )( ) | |

make products

pick	1	2	3	...		n-1	n	product	
	x	x	x	...		x	x	x^n	$\binom{n}{n}$ x, $\binom{n}{0}$ y
1y	x	x	x	...		x	y	$x^{n-1}y$	
	x	x	x	...	x	y	x	$x^{n-1}y$	
	x	x	x	...	y	x	x	$x^{n-1}y$	$\binom{n}{1} = \binom{n}{n-1}$ y, x
	⋮								
	y	x	x	...	x	x	x	$x^{n-1}y$	
2y	x	x	x	...		y	y	$x^{n-2}y^2$	$\binom{n}{n-2} = \binom{n}{2}$ x, y

$xxx \cdots yxy$ | $x^{n-2}y^2$

$xxx \cdots xyyx$ | $x^{n-2}y^2$

$xxyyx \cdots xyx$ | $x^{n-2}y^2$

$yyxxx \cdots x$ | $x^{n-2}y^2$

$3y$ $xxx \cdots xyyy$ | $x^{n-3}y^3$ $\binom{n}{3}$ y $\binom{n}{n-3}$ x

$xxx \cdots yxyy$ | $x^{n-3}y^3$

$yy \cdots xxxy$ | $x^{n-3}y^3$

$yyyxx \cdots x$ | $x^{n-3}y^3$

$4y$ $xxx \cdots xyyyy$ | $x^{n-4}\cdot y^4$

$\vdots$

$\vdots$

$2x$ $xxyyy \cdots y$ | x^2y^{n-2}

$xyxyy \cdots y$ | x^2y^{n-2}

$yy \cdots yxx$ | x^2y^{n-2}

2^n total product lines

1x	xyyy· — — — y	x^1y^{n-1}
	yxyy· — — — y	x^1y^{n-1}
	⋮	
	yyy· — — — yx	x^1y^{n-1}
0x	yyy· — — — y	y^n

times $\binom{n}{1} = \binom{n}{n-1}$
x y

$\binom{n}{0} = \binom{n}{n}$
x y

General "k-"x" "n-k" y $\boxed{x^k y^{n-k}}$

choose x from k ()

choose y from the other n-k ()

How many times do I see $x^k y^{n-k}$ term? $\binom{n}{k} = \binom{n}{n-k}$
x y

Sum it up

informal proved

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Later this term: proof by induction over n.

$$(x+y)^2 = x^2 + 2xy + y^2 \xRightarrow{2\to 3} (x+y)^3 = (x+y)^2(x+y)$$
$$= (x^2+2xy+y^2)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$$

Binomial theorem: $(x+y)^n$

$(x+y)^0 = 1$

$(x+y)^1 = x + y$

$\boxed{(x+y)^2} = x^2 + 2xy + y^2 = (x+y)(x+y)$

choose x or y | choose x or y

	x	y
x	x^2	xy
y	yx	y^2

$(x+y)(x+y)$

$(x+y)^3 = (x+y)(x+y)(x+y)$

0 y's ←	F	x	x	→	x^2
1 y ←	O	x	y	→	xy
	I	y	x	→	yx
2 y's ←	L	y	y	→	y^2

$\Rightarrow x^2 + 2xy + y^2$

Q: How many y's do you choose → dictates term, e.g., xy^2 vs. x^2y

How many ways to do so? → dictates coefficient in front of term.

A:

	# ways	term
0 y's	1	x^3
1 y	$3 = \binom{3}{1}$	x^2y
2 y's	$3 = \binom{3}{2}$	xy^2
3 y's	1	y^3

$\Rightarrow$

1 3 3 1

$\boxed{x^3 + 3x^2y + 3xy^2 + y^3}$

$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$

2 y's $\Rightarrow \binom{4}{2}$ options to get

How many y's ?

	# ways	term
0	$1 = \binom{4}{0}$	x^4
1	$4 = \binom{4}{1}$	$x^3 y$
y 2	$6 = \binom{4}{2}$	$x^2 y^2$
3	$4 = \binom{4}{3}$	$x y^3$
y 4	$1 = \binom{4}{4}$	y^4

1 4 6 4 1

$\Rightarrow x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^n = (x+y)(x+y) \cdots (x+y)$ ← n terms

# y's	# ways	term
0	$\binom{n}{0}$	x^n
1	$\binom{n}{1}$	$x^{n-1}y$
2	$\binom{n}{2}$	$x^{n-2}y^2$
$\vdots$	$\vdots$	$\vdots$
n	$\binom{n}{n}$	y^n

$\Rightarrow$

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

$$= \boxed{\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j}$$

index by j instead of k

Application:

$$(x + 2y^{-2})^6$$

Q: What is the term that includes y^{-8}?

$$= \overbrace{(x+2y^{-2})(x+2y^{-2}) \cdots (x+2y^{-2})}^{6}$$

$\Rightarrow$ to get y^{-8}, need to expand by $2y^{-2}$ $\boxed{4 \text{ times}}$

y term

$$\binom{6}{4} x^2 (2y^{-2})^4$$

$$= \binom{6}{2} x^2 (2y^{-2})^4$$

$$= \frac{6 \cdot 5}{\cancel{2} \cdot 1} x^2 \cdot 2^{\cancel{4}\,3} y^{-8}$$

$$= 240\, x^2 y^{-8}$$

✓

$x=1 \quad y=1$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad =1$$

$$\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k}} \iff 2^n = \sum_{k=0}^{n} (\#\text{subsets of size } k)$$

$|set| = n$

$\binom{n}{k}$: # subsets of size k

$|size| = 0$

$|size| = 1$

$2^n =$

$|size| = n$

$2^n = \#$ all subsets

$2^n = |\mathcal{P}(set)|$ powerset size

$x=1 \quad y=-1$

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$$

$$(1-1)^n$$

$$0 = +\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots (-1)^n \binom{n}{n}$$

alternate the sign $\Rightarrow$ sum of binomial coef with alternate signs $\pm$

$= 0$

$$(x+y)^4 = (x+y)(x+y)^3$$

$$= (x+y)(x^3+3x^2y+3xy^2+y^3)$$

$n=4$

$$\begin{array}{lccccc} = & x^4 + & 3x^3y + & 3x^2y^2 + & xy^3 & \\ + & & x^3y + & 3x^2y^2 + & 3xy^3 + & y^4 \\ \hline & x^4 + & 4x^3y + & 6x^2y^2 + & 4xy^3 + & y^4 \end{array}$$

$n=3$:

$$\begin{array}{ccccc} \cancel{1} & 3 & 3 & 1 & \\ & 1 & 3 & 3 & 1 \\ \hline 1 & 4 & 6 & 4 & 1 \end{array}$$

$\binom{3}{2}$ $\binom{3}{1}$ $\binom{4}{2}$

$$\boxed{\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}}$$

exercise

- with combinations
- with !!

$$\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$$

need 2 y's

expand by x in first term; need 2 y's from 2nd term

expand by y in first term, need 1 y in 2nd term

Choose 3 out of 8 people

$$\binom{8}{3} = \binom{7}{3} + \binom{7}{2}$$

Pascal's Triangle

$(x+y)^n$

$j = \#\, y\text{'s}$

	0	1	2	3	4 ...
$n=0$	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1
5					
⋮					

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$$

$(x+y)^0$: 1

$(x+y)$: 1 1

$x^2+2xy+y^2$: 1 2 1 $(x+y)^2$

$x^3+3x^2y+3xy^2+y^3$: 1 3 3 1

x^4: 1 4 6 4 1

$x^4 \quad 4x^3y \quad 6x^2y^2 \quad 4xy^3 \quad y^4$

n=5: 1 5 10 10 5 1

next row coef for $(x+y)^5$

n=6: 1 6 15 20 15 6 1

$(x+y)^6$: $x^6 + 6x^5y + 15x^4y^2 \ldots$

Applications & Consequences

① What is $\boxed{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n}$

Consider $S = \{a, b, c\}$

$P(S)$

0	$\emptyset$	$1 = \binom{3}{0}$	
1	$\{a\}\ \{b\}\ \{c\}$	$3 = \binom{3}{1}$	$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$
2	$\{a,b\}\ \{b,c\}\ \{a,c\}$	$3 = \binom{3}{2}$	
3	$\{a,b,c\}$	$1 = \binom{3}{3}$	
		8	

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

1 1

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot 1^{n-j} \cdot 1^j = \sum_{j=0}^{n} \binom{n}{j}$$

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1331$$
$$11^4 = 14641$$

$$11^n = (\overset{x}{1} + \overset{y}{10})^n = \sum_{j=0}^{n} \binom{n}{j} \cdot 1^{n-j} \cdot 10^j$$
$$= \sum_{j=0}^{n} \binom{n}{j} \cdot 10^j$$
$$= \binom{n}{n} \cdot 10^n + \binom{n}{n-1} 10^{n-1} \cdots \binom{n}{0} \cdot 10$$

$$11^3 = (1+10)^3 = \binom{3}{3} \cdot 10^3 + \binom{3}{2} 10^2 + \binom{3}{1} \cdot 10^1 + \binom{3}{0} \cdot 10^0$$
$$= 1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10 + 1$$
$$= 1331$$

1

1 1

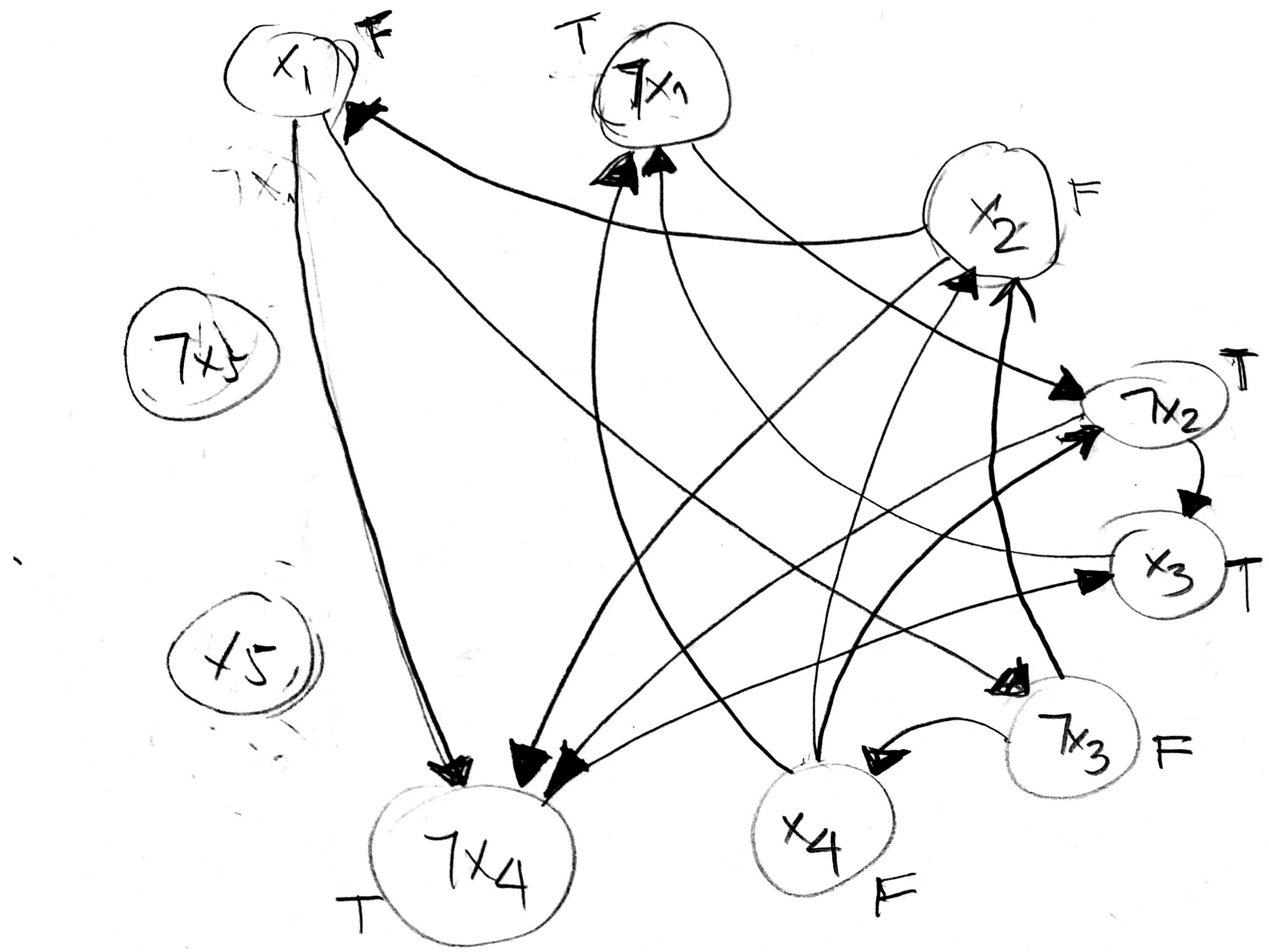
1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1



$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_4)$

$x_2 \Rightarrow x_1$	$x_1 \Rightarrow \neg x_3$	$\neg x_3 \Rightarrow x_4$	$x_2 \Rightarrow \neg x_4$	$\neg x_2 \Rightarrow \neg x_4$
$\neg x_1 \Rightarrow \neg x_2$	$x_3 \Rightarrow \neg x_1$	$\neg x_4 \Rightarrow x_3$	$x_4 \Rightarrow \neg x_2$	$x_4 \Rightarrow x_2$

$(\neg x_1 \vee \neg x_4) \wedge (x_2 \vee x_3)$

$x_1 \Rightarrow \neg x_4$	$\neg x_2 \Rightarrow x_3$
$x_4 \Rightarrow \neg x_1$	$\neg x_3 \Rightarrow x_2$

does it give a contradiction?

$\neg x_2 \Rightarrow x_1 \Rightarrow x_2 \Rightarrow x_3 \Rightarrow \begin{cases} x_3 = T \\ \neg x_3 = F \end{cases}$

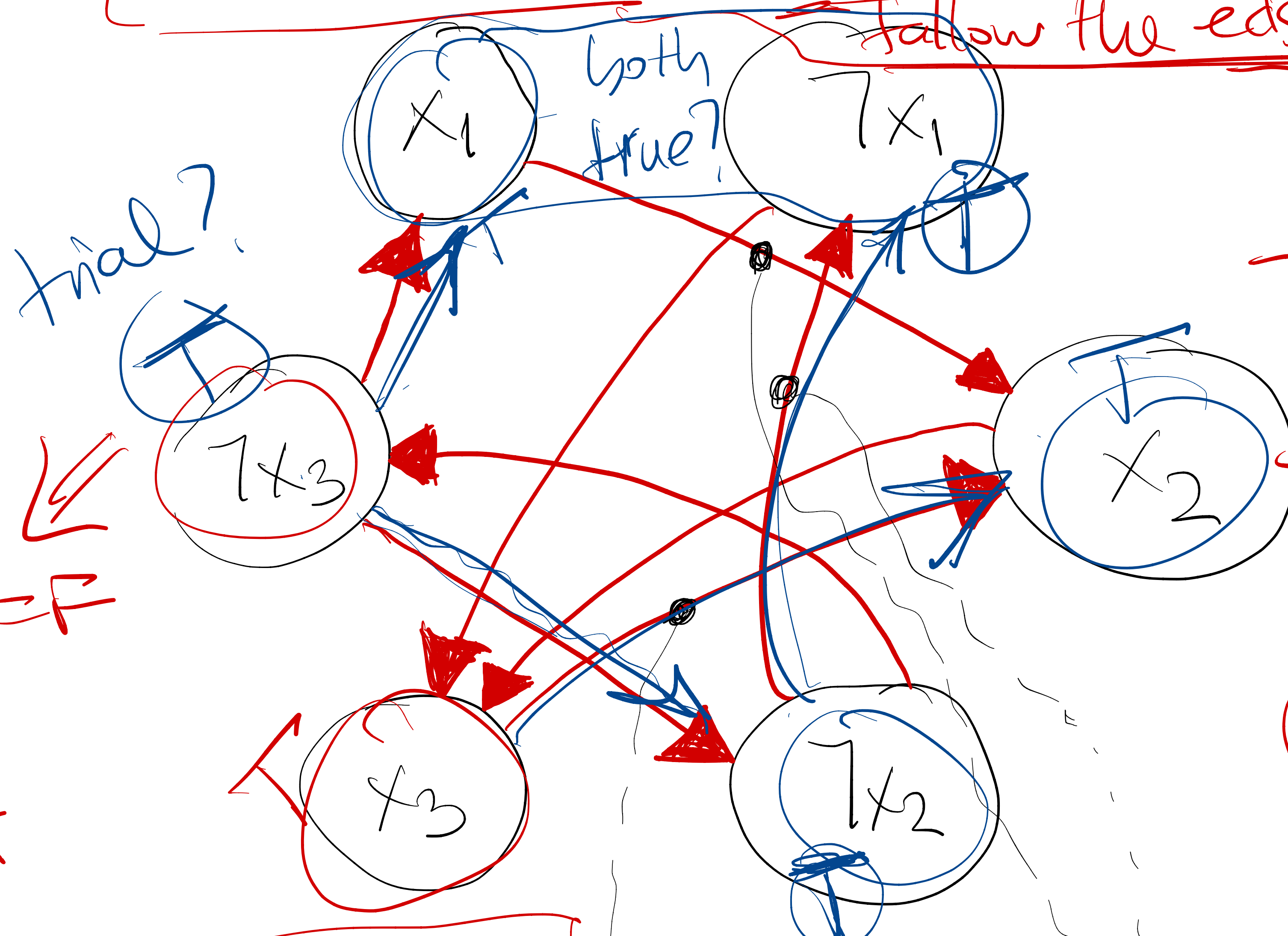
Trial and error: start with var = T
follow the edges

1) input formula
each clause $(x \vee y)$

2) transform each clause
into 2 implications

3) Graph:
nodes: all literals
and all $\neg$literals

4) Make sure all
implications are true.



$\neg x_3 = F$

$x_3 = T$

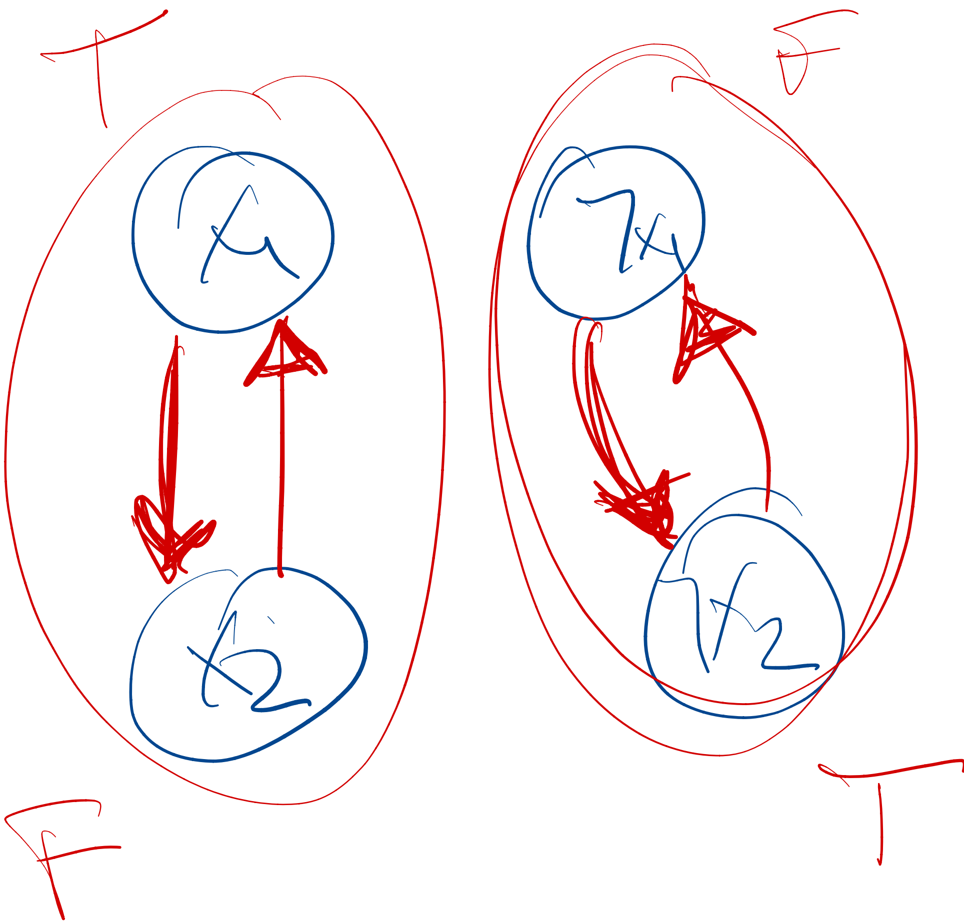
$(x_2 \vee \neg x_3) \wedge (x_3 \vee x_1) \wedge (x_2 \vee \neg x_1) \wedge (x_3 \vee \neg x_2)$

$\boxed{x_3 \Rightarrow x_2}$	$\neg x_3 \Rightarrow x_1$	$\boxed{\neg x_2 \Rightarrow \neg x_1}$	$\neg x_3 \Rightarrow \neg x_2$
$\neg x_2 \Rightarrow \neg x_3$	$\neg x_1 \Rightarrow x_3$	$\boxed{x_1 \Rightarrow x_2}$	$x_2 \Rightarrow \cancel{\neg} x_3$

$$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$$

$\neg x_1 \Rightarrow \neg x_2$ $\qquad$ $x_1 \Rightarrow x_2$

$x_2 \Rightarrow x_1$ $\qquad$ $\neg x_2 \Rightarrow \neg x_1$



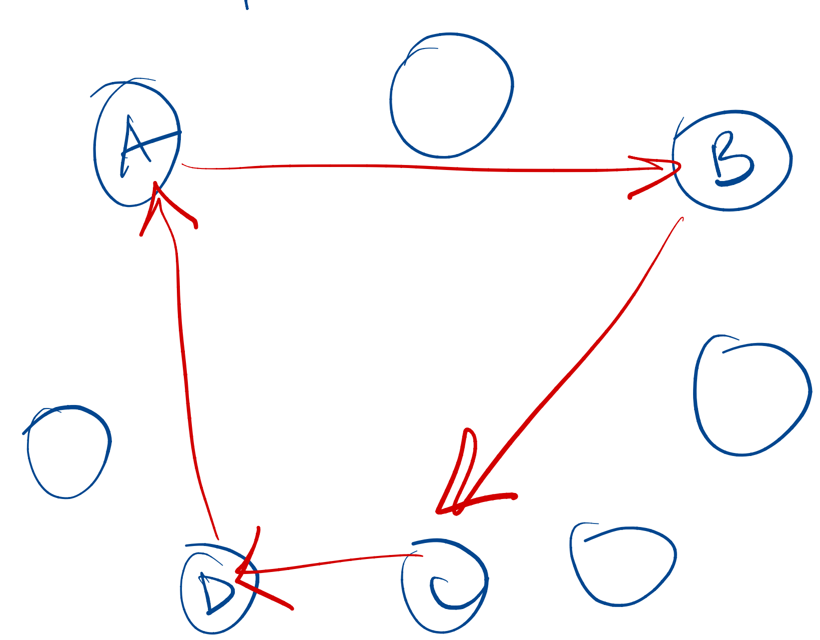
$x_1, x_2 \Rightarrow$ same val

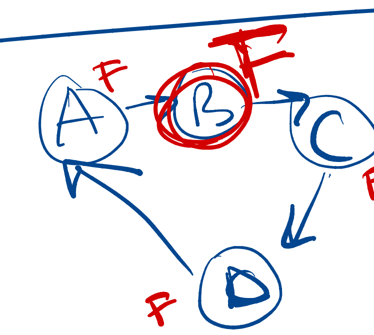
$\neg x_1, \neg x_2 \Rightarrow$ same value

$\boxed{X \vee Y} \equiv \boxed{\neg X \Rightarrow Y} \equiv \boxed{\neg Y \Rightarrow X}$

X	Y	$\neg X$	$\neg Y$	$X \vee Y$	$\neg X \Rightarrow Y$	$\neg Y \Rightarrow X$
0	0	1	1	0	0	0
0	1	1	0	1	1	1
1	0	0	1	1	1	1
1	1	0	0	1	1	1

Loop in the graph

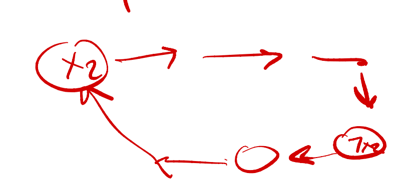




A, B, C, D must be $\left\{ \begin{array}{l} \text{all T} \\ \text{or} \\ \text{all F} \end{array} \right.$

Consequence:

If x_2 and $\neg x_2$ are both part of a loop?



part A, Satisfiability Intro [easy]. A boolean formula is satisfiable if there exists some variable assignment that makes the formula evaluate to true. Namely, a boolean formula is satisfiable if there is some row of the truth table that comes out true. Determining whether an arbitrary boolean formula is satisfiable is called the *Satisfiability Problem.* There is no known efficient solution to this problem, in fact, an efficient solution would earn you a million dollar prize. While this is hard problem in computer science, not all instances of the problem are hard, in fact, determining satisfiability for some types of boolean formulae is easy.

i. First, let's consider why this would be hard. If you knew nothing about a given boolean formula other than that it had n variables, how large is the truth table you would need to construct? Please indicate the number of columns and rows as a function of n

ii. Now consider the following 100 variable formula.

BIG TT

$$x_1 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4) \wedge \ldots \wedge (\neg x_{99} \vee x_{100})$$

Without constructing a truth table, how many satisfying assignments does this formula have, explain your answer.

every clause 3 literals, ex. $(x_1 \wedge x_5 \wedge \neg x_{72})$

not CNF

iii. Now consider an arbitrary 3-DNF formula with 100 variables and 200 clauses. 3-DNF means that the formula is in disjunctive normal form and each clause has three literals. (A literal is the instantiation of the variable in the formula, so for x, $\neg x$ or x.) An example might be something like:

$$(\neg x_1 \wedge x_3 \wedge x_{10}) \vee (\neg x_3 \wedge x_{15} \wedge \neg x_{84}) \vee (x_{17} \wedge \neg x_{37} \wedge x_{48}) \vee \ldots \vee (\neg x_{87} \wedge \neg x_{95} \wedge x_{100})$$

What is the largest size truth table needed to solve this problem. What is the maximum number of such truth tables needed to determine satisfiabilty.

TRICKY

$x_1 \quad x_3 \quad x_{10} \quad \vee$ $\qquad$ $x_{17} \quad x_{37} \quad x_{48}$

2 options on first clause

8

1) ∃ ⇒ true

2) ∀ ⇒ false

8 $\qquad$ 8

8 rows per TT, max 200 TTs ⇒ correct?

– overlap?. x_2 or $\neg x_2$ in multiple clauses.

satisfiability = "make it true" DNF ⟺ one clause has to be true.