E-unifiability via Narrowing

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Menu

- Introduction to E-unifiability
- Quick survey of decidability results
- Term Rewriting Systems
- Narrowing
- Narrowing bounds
- Applications

Introduction

Signature Σ	$+, 0$
Variables X	$x, y, ...$
Terms $T(\Sigma, X)$	$x + (y + 0), 0 + 0$

Equational theory E

e.g. $AC = \{x+(y+z) \approx (x+y)+z, x+y \approx y+x\}$

E-unifiability

Given E-unification problem

$\Pi = \{s_1 =^?_E t_1, ..., s_n =^?_E t_n\}$

$\exists$ substitution θ such that

$\theta s_1 =_E \theta t_1, ..., \theta s_n =_E \theta t_n$?

Motivation

Theorem Proving (E-unifiability $\Leftrightarrow Th_+(E)$)

Examples

Let $\Sigma := \{f(.,.), g(.), +, a, b\}$

$$f(x,b) =^?_\emptyset f(a,y) \text{ yes:}$$
$$\theta(x) := a, \theta(y) := b$$
$$\Downarrow$$
$$\theta(f(x,b)) = f(a,b) = \theta(f(a,y))$$

$$x =^?_\emptyset g(x) \text{ no}$$

$$x =^?_{\{g(x)\approx g(g(x))\}} g(x) \text{ yes:}$$
$$\theta(x) := g(x)$$
$$\Downarrow$$
$$\theta(x) = g(x) =_E g(g(x)) = \theta(g(x))$$

$$(a+x)+b =^?_{AC} (a+b)+(a+b) \text{ yes:}$$
$$\theta(x) := a+b$$
$$\Downarrow$$
$$\theta(a+x)+b = (a+(a+b))+b$$
$$=_{AC} (a+b)+(a+b) = \theta((a+b)+(a+b))$$

Complexity

$\emptyset$-unifiability $\in P$

AC-unifiability $\in NP - complete$

...

Hilbert's 10th-unifiability undecidable

General solutions

[Nieuwenhuis98] In E variables at depth $\leq 1 \Rightarrow$ E-unifiability $\in NP$

[Baader,Schulz96] Combination techniques:

$$E = E_1 \cup E_2, \Sigma(E_1) \cap \Sigma(E_2) = \emptyset$$
$$\Downarrow$$
$$E\text{-unifiability} \Leftrightarrow E_1\text{-unifiability, } E_2\text{-unifiability}$$

Concrete Problem

MPA^{+}-unifiability

A is a set of actions

$$\begin{array}{rcll}
(x+y)+z & \approx & x+(y+z) & \\
x+y & \approx & y+x & \\
x+x & \approx & x & \\
x+\delta & \approx & x & \\
a(a^{+}x+x) & \approx & a^{+}x & \forall a \in A \\
a^{+}(a^{+}x+x) & \approx & a^{+}x & \forall a \in A
\end{array}$$

Complete axiomatization of bisimulation equivalence for process algebra with prefix iteration [Fokkink94]

[Guo,Narendran,Shukla98] Unifiability in process algebras (MPA)

[Intrigila,Venturini-Zilli,Viola00] Survey of unifiability in process algebras

No general result is applicable!

Term Rewriting

A Term Rewriting System T $\{x+x \to x\}$

Rewrite Relation $\to_T$ $f(f(y)+f(y)) \to_T f(y)$

T convergent (confluent and terminating) $\Rightarrow$ every term t has unique normal form $t\downarrow_T$

E equivalent to a convergent T

$$\theta s =_E \theta t \Leftrightarrow \theta s \leftrightarrow_T \theta t$$
$$\Updownarrow$$
$$\theta s \to_T \ldots \to_T \theta s \downarrow_T = \theta t \downarrow_T \leftarrow_T \ldots \leftarrow_T \theta t$$

Narrowing

$$\theta s \to_T s' \to_T s'' \to_T \ldots$$
$$\Updownarrow$$
$$s \rightsquigarrow_T \overline{s} \rightsquigarrow_T \overline{s}'' \rightsquigarrow_T \ldots$$

To solve $s =_E^? t$ guess $\rightsquigarrow$ reductions

Mostly semi-decision procedure. [Hullot80] gives some decidability results (no complexity)

Narrowing Bound

H narrowing bound:

$\Pi := \{s =^?_E t\}$ unifiable $\Rightarrow \exists \theta : \theta s {\downarrow}_T^{H(\Pi)}, \theta t {\downarrow}_T^{H(\Pi)}$

$\exists$ Polynomial narrowing bound

$\Downarrow$

E-unifiability $\in NP$

Nice: $E = T = \{s_i \to t_i\}$

$\forall i, t_i \leq s_i \Rightarrow E$-unifiability $\in NP$

Applications

$\{(x + x \approx x\}$-unifiability $\in NP$ (also $NP - hard$)

$\{(x * y) * (y * z) \approx y\}$-unifiability $\in NP$

$\{x * (x \backslash y) \approx y, (x * y)/y \approx x,$
$(x/y) * y \approx x, (x/y) \backslash x \approx y,$
$x \backslash (x * y) \approx y, x/(y \backslash z) \approx y\}$-unifiability $\in NP$

AC-case

AC cannot be oriented $\quad x + y \to_T y + x \to_T \ldots$

Rewriting modulo AC: $\to_{T/AC} \Leftrightarrow =_{AC} \to_T =_{AC}$

Narrowing modulo AC (extensions) [Plotkin72], [Peterson,Stickel81], [Vigneron98]

$$\begin{gathered}\exists \text{ Polynomial narrowing bound} \\ \Downarrow \\ E\text{-unifiability} \in NP\end{gathered}$$

(but nice result about subterms is false)

Applications

$$
\begin{array}{rcll}
(x+y)+z & \approx & x+(y+z) & \\
x+y & \approx & y+x & \\
x+x & \approx & x & \\
x+\delta & \approx & x & \\
a(a^+x+x) & \approx & a^+x & \forall a \in A \\
a^+(a^+x+x) & \approx & a^+x & \forall a \in A
\end{array}
$$

Convergent AC-TRS [Fokkink94]:

$$
\begin{array}{rcll}
x+x & \to & x & \\
x+\delta & \to & x & \\
a(a^+x+x) & \to & a^+x & \forall a \in A \\
a^+(a^+x+x) & \to & a^+x & \forall a \in A \\
a(a^+\delta) & \to & a^+\delta & \forall a \in A \\
a^+(a^+\delta) & \to & a^+\delta & \forall a \in A
\end{array}
$$

Has a polynomial narrowing bound!

(Also $AC \cup \{x+x \approx x\}$
and $AC \cup \{x+x \approx x, x+\delta \approx x\}$)