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CS G399: Algorithmic Power Tools I — Scribe: Jian Wen

Lecture Outline:

- Primal-dual schema
- Primal-dual for set cover

This lecture describes the primal-dual schema, and also an application of this method to design approximation algorithms.

1 Primal-dual schema

In a previous lecture we have introduced the concept of duality for LP problems. A common form of the LP problem with its duality is showed below:

$$\begin{array}{rl} & \textit{Primal} \\ & \min \sum_i c_i x_i \\ s.t. & \sum_i a_{ij} x_i \geq b_j \\ & x_i \geq 0 \end{array} \qquad \begin{array}{rl} & \textit{Duality} \\ & \max \sum_j b_j y_j \\ s.t. & \sum_j a_{ij} y_j \leq c_i \\ & y_j \geq 0 \end{array}$$

From the principle of duality, a feasible solution of the dual in fact sets a lower bound for the primal problem. We here use a concept of **complementary slackness** to help us illustrate the connection between the two solutions of the primal and duality problem:

Definition 1. The complementary slackness condition is as follows:

$$\begin{aligned} \textit{Primal} &: \; x_i > 0 \Rightarrow \sum_j a_{ij} y_j = c_i \\ \textit{Dual} &: \; y_j > 0 \Rightarrow \sum_i a_{ij} x_i = b_j \end{aligned}$$

Then the following theorem will show this connection:

Theorem 1. *If (x, y) satisfies* **complementary slackness***, then x and y are optimal solutions for primal and dual problems, respectively.*

Proof: From the forms of the primal and duality, if complementary slackness is satisfied, we will have

$$\begin{aligned} \sum_i c_i x_i &= \sum_i (\sum_j a_{ij} y_j) * x_i \\ \sum_j b_j y_j &= \sum_j (\sum_i a_{ij} x_i) * y_j \end{aligned}$$

It is easy to see that the RHS of the two inequations are equal. So we have

$$\sum_i c_i x_i = \sum_j b_j y_j$$

implying that x and y are both optimal by duality.

□

So we can use the complementary slackness to solve LP problems for an optimal solution. While for integer LP, since it is NP-Hard, we need to get an approximately optimal solution in polynomial time. In order to apply the complementary slackness theorem into the approximate situation, we define a **relaxed slackness** as follows:

Definition 2.

$$\begin{aligned} Primal &: \quad x_i > 0 \Rightarrow \frac{c_i}{\alpha} \leq \sum_j a_{ij} y_j \leq c_i \\ Dual &: \quad y_j > 0 \Rightarrow b_j \geq \sum_i a_{ij} x_i \geq \beta b_j \end{aligned}$$

And accordingly, we will fit 1 into the approximate situation with the description below:

Theorem 2. *If (x, y) satisfies* **relaxed complementary slackness**, *then x and y are $\alpha\beta$-optimal for both primal and dual problem.*

Proof: Based on the definition of relaxed complementary slackness, we will have

$$\sum_i c_i x_i \leq \alpha \sum_i (\sum_j a_{ij} y_j) * x_i \tag{1}$$

$$\sum_j b_j y_j \geq \frac{1}{\beta} \sum_j (\sum_i a_{ij} x_i) * y_j \tag{2}$$

Then we can get

$$\frac{\text{cost of x}}{\text{cost of y}} = \frac{\sum_i c_i x_i}{\sum_j b_j y_j} \leq \alpha\beta$$

which indicates that the duality yields a $\alpha\beta$-approximation solution. □

Based on the property of the relaxed complementary slackness, the primal-dual schema for LP solution can be done in an iterative process: start with a dual feasible solution, then we try to improve this dual problem, until the improved solution reaches satisfied the relaxed complementary slack conditions.

2 Primal-dual for set cover

In this section we will apply the primal-dual schema to the set cover problem. Recall the primal problem of set cover with universe U of elements and collection C of sets, and its dual:

$$\begin{array}{rlrl} & \textit{Primal} & & \textit{Duality} \\ & \min \sum_{S \in C} c(S) x_S & & \max \sum_{e \in U} y_e \\ s.t. & \sum_{S:\ e \in S} x_S \geq 1,\ \forall e \in U & s.t. & \sum_{e \in s} y_e \leq c(S),\ \forall S \in C \\ & x_j \geq 0,\ \forall j & & y_e \geq 0,\ \forall e \end{array}$$

As mentioned above, we will take an iterative process to get the approximate solution. First of all, we start with a primal unfeasible but dual feasible solution

$$x = 0,\ y = 0$$

The iterative algorithm is as following:

- While any element is uncovered
  1. Pick any uncovered elements e
  2. Increase y_e until some set S become tight, i.e., $\sum_{e \in S} y_e = c(S)$.
  3. Add S to solution.

According to the definition of the relaxed complementary slackness, we will have

$$x_S > 0 \Rightarrow \sum_{e \in S} y_e = c(S) \qquad \alpha = 1 \tag{3}$$

$$y_e > 0 \Rightarrow \sum_{S:\ e \in S} x_S \leq f_e \qquad \beta = \max_e f_e \tag{4}$$

Here we let f_e be the number of the sets that contain e. So from the above analysis we see that the primal-dual algorithm will get a f-approximate solution, where f is the maximum frequency of any element.