College of Computer & Information Science
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- Introduction: random walks in directed graphs
- Random walks in d-regular undirected graphs

1 Introduction to random walks

Let G be a d-regular directed graph (which is any directed graph in which the in-degree and out-degree of every vertex is d). Let A denote its adjacency matrix, with $A_{ij} = 1$ whenever there is an edge from i to j. A random walk on G is a process that starts at time 0 at an arbitrary vertex and proceeds as follows: if the walk is at vertex i at time t, then it is at vertex j at time $t + 1$ where j is chosen uniformly at random from all the out-neighbors of i. We can capture the random walk by the matrix $M = A/d$, which we call the *random walk matrix*.

A vector x is referred to as a probability vector is $\sum_i x(i) = 1$. The location of the walk at time t can be given by the probability vector $p_t = p_0 M^t$, where p_0 is the unit vector with $p_0(i) = 1$ if the walk starts at i and 0 otherwise. We say that x is a stationary distribution if $xM = x$. Some immediate questions are:

- Do stationary distributions always exist? It is immediate that $u = (1/n, \ldots, 1/n)$ is a stationary distribution of a random walk for every d-regular graph. More generally, finite Markov chain has a stationary distribution.
- Is the stationary distribution unique? In general, any finite irreducible ergodic Markov chain has a unique stationary distribution.
- Does a random walk always converge to a stationary distribution? If it does, how long does it take?

We now study the convergence of a random walk on G to the stationary distribution u. Let $\lambda(G)$ be defined as follows:

$$\lambda(G) = \max_{x \perp u} \frac{\|xM\|}{\|x\|}.$$

Lemma 1. *For any initial probability distribution π, we have*

$$\|\pi M^k - u\| \leq \lambda(G)^k.$$

Proof: Since $\|xM\| \leq \lambda(G)\|x\|$ for any $x \perp u$, it follows that $\|(\pi - u)M\| \leq \lambda(G)\|\pi - u\|$. Since $(\pi - u)M^t \perp u$ for all t, it follows that $\|(\pi - u)M^t\| \leq \lambda(G)^t \|\pi - u\|$. But $uM^t = u$ and $\|\pi - u\| \leq 1$, yielding the desired claim. □

From the above lemma, it is clear that a random walk will converge to the stationary distribution if $\lambda(G) < 1$; smaller the value of $\lambda(G)$, the faster it will converge. In particular, in $\ln(n/\varepsilon)/(1-\lambda(G))$ steps, every entry of πM^k will be at least $(1-\varepsilon)/n$. Is there a good upper bound on $\lambda(G)$ be? We will see that for undirected nonbipartite graphs $\lambda(G) \leq 1 - \Omega(1/n^3)$, which implies that a random walk on nonbipartite graphs converges to the uniform distribution in a polynomial number of steps.

2 Random walks in d-regular undirected graphs

For a d-regular undirected graph M is symmetric. This implies that it has n real eigenvalues and corresponding real eigenvectors. Note that u is an eigenvector with eigenvalue 1. One can see that every other eigenvalue is less than 1; and in fact, every eigenvalue of M has absolute value at most 1. Let $1 = \mu_1 \geq |\mu_2| \geq \ldots \geq |\mu_n|$ denote the n eigenvalues of M and let $u = v_1, v_2, \ldots, v_n$ denote the corresponding eigenvectors.

Lemma 2. *For every probability distribution π, we have*

$$\|\pi M^k - u\| \leq \mu_2^k \|\pi - u\|.$$

Proof: We write $\pi - u$ (which is orthogonal to u) as a linear combination of the eigenvectors $v_2, \ldots, v_n$:

$$\pi - u = \sum_{i=2}^{n} c_i v_i$$

We now derive the desired inequality as follows.

$$\begin{aligned}
\|\pi M^k - u\| &= \|(\pi - u)M^k\| \\
&= \|\sum_{i=2}^{n} c_i v_i M^k\| \\
&= \|\sum_{i=2}^{n} c_i \mu_i^k v_i\| \\
&\leq |\mu_2|^k \|\sum_{i=2}^{n} c_i v_i\| \\
&= |\mu_2|^k \|\pi - u\|
\end{aligned}$$

Another way to see it is to show $\lambda(G) = |\mu_2|$, whose proof is essentially embedded above. Take any $x \perp u$. Then, it follows that $x = \sum_{i=2}^{n} a_i v_i$ for some coefficients $a_2, \ldots, a_n$. Then we have

$$\begin{aligned}
\|xM\|^2 &= \|\sum_{i=2}^{n} a_i v_i M\|^2 \\
&= \|\sum_{i=2}^{n} a_i \mu_i v_i\|^2 \\
&\leq |\mu_2|^2 \|\sum_{i=2}^{n} a_i v_i\|^2 \\
&= |\mu_2|^2 \|x\|.
\end{aligned}$$

We have thus established that $\lambda(G) \leq |\mu_2|$. To see the other direction, take $x = v_2$ and we obtain that $\|xM\|/\|x\| = |\mu_2|$, thus showing that $\lambda(G) \geq |\mu_2|$. □

This brings us to the question: what is $|\mu_2|$ for an arbitrary undirected graph G. Here are some things we can show.

1. All the eigenvalues of M have absolute value at most 1. One way to see this is that $M = I - L/d$, where L is the Laplacian of G. Thus, the eigenvalues of M are $(1 - \lambda_i/d)$, where λ_is are the eigenvalues of the Laplacian. By Courant-Fischer, the largest eigenvalue is given by

$$\max_{x \neq 0} \frac{x^T M x}{x^T x} = 1 - \min_{x \neq 0} \frac{x^T L x}{d x^T x} \leq 1.$$

   By Courant-Fischer, the smallest eigenvalue is given by

$$\min_{x \neq 0} \frac{x^T M x}{x^T x} = 1 - \max_{x \neq 0} \frac{x^T L x}{d x^T x} = 1 - \max_{x \neq 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_i x_i^2} \geq 1 - \max_{x \neq 0} \frac{\sum_{(i,j) \in E} 2(x_i^2 + x_j^2)}{d \sum_i x_i^2} = -1.$$

2. We can also show that M has an eigenvalue of -1 if and only if M is bipartite.

3. If G is connected and nonbipartite, then all eigenvalues other than the largest (which is 1) have magnitude strictly less than 1. This follows from the fact that the null-space of the Laplacian of a connected graph has dimension 1.

4. If G is connected and nonbipartite, then $|\mu_2|$ is at most $1 - 1/\text{poly}(n)$. In particular, we show that $|\mu_2|$ is at most $1 - 1/(4dn^3)$. We show it in two parts: first that the second largest eigenvalue is at most $1 - 1/(4dn^3)$. Second, we argue that the smallest eigenvalue is at least $-1 + 1/(4dn^3)$.

   We first make the following observation:

$$\begin{aligned} x^T M x &= \sum_i \Big(\sum_{(i,j) \in E} x_i x_j / d \Big) \\ &= \sum_{(i,j) \in E} 2 x_i x_j / d \\ &= \sum_{(i,j) \in E} (x_i^2 + x_j^2 - (x_i - x_j)^2)/d \\ &= 1 - \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2. \end{aligned}$$

   Consider the second largest eigenvalue, which equals

$$\max_{x \perp \mathbf{1}, \|x\|=1} x^T M x = 1 - \min_{x \perp \mathbf{1}, \|x\|=1} \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2.$$

   Sort the x_i values in nondecreasing order $x_1 \leq x_2 \leq \ldots \leq x_n$. Since $\|x\| = 1$ and $\sum_i x_i = 0$, it follows that $x_1 \leq 0 \leq x_n$ and $|x_1 - x_n| \geq 1/\sqrt{n}$. Therefore, there exists at least one i such that $|x_i - x_{i-1}| \geq 1/n^{3/2}$. Since the graph is connected, there exists an edge (i, j) with $|x_i - x_j| \geq 1/n^{3/2}$, which implies that the second largest eigenvalue is at most $1 - 1/(dn^3)$.

   We now argue that the largest eigenvalue is at least $-1 + 1/(4dn^3)$ if the graph is nonbipartite and connected. Note that $x^T Mx$ can also be written as follows.

$$\begin{aligned} x^T Mx &= \sum_i (\sum_{(i,j)\in E} x_i x_j / d) \\ &= \sum_{(i,j)\in E} 2x_i x_j / d \\ &= \sum_{(i,j)\in E} ((x_i + x_j)^2 - x_i^2 - x_j^2)/d \\ &= -1 + \sum_{(i,j)\in E} (x_i + x_j)^2 / d. \end{aligned}$$

Therefore, the largest eigenvalue equals

$$\min_{x\perp \mathbf{1}, \|x\|=1} x^T Mx = -1 + \min_{x\perp \mathbf{1}, \|x\|=1} \frac{1}{d} \sum_{(i,j)\in E} (x_i + x_j)^2.$$

Without loss of generality, assume $|x_1| \geq |x_n|$, so $x_1 \leq -1/\sqrt{n}$. If there is any vertex k such that $|x_k| \leq 1/(2\sqrt{n})$, then consider a shortest path from 1 to k: the sum of $(x_i + x_j)^2$ along the edges (i,j) of this path is at least $\sum_{(i,j)\in E} (|x_i| - |x_j|)^2$, which is at least $1/(4n^3)$. Otherwise, all vertices have $|x_k| \geq 1/(2\sqrt{n})$. Now, we consider two cases. First, there is an edge (i,j) with x_i and x_j having the same sign. In this case $(x_i + x_j)^2 \geq 1/n$, and we are done. Otherwise, all edges are from vertices with positive x_i to vertices with negative x_i; but this can only happen if the graph is bipartite.