Problem Set 5 (due Monday, November 10)

1. (10 points) Single-variable linear program

Exercise 29.5-9 of text.

2. (10 points) LP for single-source shortest paths

Exercise 29.2-3 of text.

3. (20 points) Complementary slackness

Problem 29-2 of text.

4. (3 + 7 = 10 points) Markov chains and LP duality

A Markov chain is a set of states (of a system) and the probability, for each pair of states, of transitioning from one state to the other. Markov chains have a wide range of applications including as models for several physical or biological processes, in economics and the social sciences, web search, and statistics.

Formally, a Markov chain is an $n \times n$ matrix P where p_{ij} denotes the probability of transitioning from state i to state j; if the system is in state i at time t, then the probability that it is in state j at time $t+1$ is p_{ij}. Clearly, P satisfies the property that the sum of the entries along any row is 1.

One nice, and very useful, property of Markov chains is the existence of a stationary distribution π over the state space, where π is an $n \times 1$ vector with π_i being the probability of the system being in state i. We say that π is stationary if the following holds: if at any time t, π gives the probability distribution for the state of the given system, then π is also the probability distribution for the state in time $t+1$.

For any Markov chain matrix, the existence of a stationary distribution π can be shown easily using LP duality, as we establish in this exercise.

(a) Consider the following LP (over variables π_i) whose constraints define the stationary distribution.

$$\begin{aligned} \min \quad & \textstyle\sum_i \pi_i \\ \left(\sum_j p_{ji}\pi_j\right) - \pi_i &= 0 \quad \forall i \\ \sum_j \pi_j &\geq 1 \\ \pi_i &\geq 0 \quad \forall i \end{aligned}$$

Show that the above LP is feasible if and only if the Markov chain has a stationary distribution.

(b) Derive the dual for the above LP. Analyze the dual and argue using duality that the primal is always feasible.