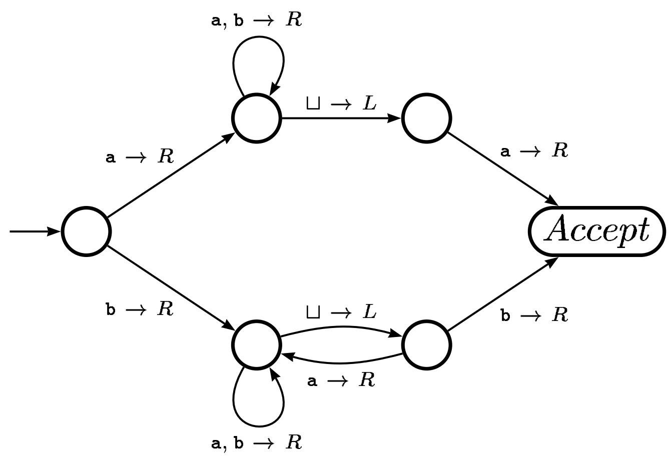
Language Deciders vs. Language Recognizers

Every TM T partitions the set of possible input strings over its input alphabet into three sets:

- ACCEPT(T) = the set of input strings for which T halts by reaching its accept state;
- REJECT(T) = the set of input strings for which T halts by reaching its reject state; and
- LOOP(T) = the set of input strings for which T never halts.

For example, consider the following TM M for strings over the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$:



It is easily checked that

- ACCEPT(M) = all strings of the form $\texttt{a}\Sigma^*\texttt{a} \cup \texttt{b}\Sigma^*\texttt{b} \cup \Sigma$;
- REJECT(M) = all strings of the form $\texttt{a}\Sigma^*\texttt{b} \cup \varepsilon$; and
- LOOP(M) = all strings of the form $\texttt{b}\Sigma^*\texttt{a}$.

Using this terminology, we can redefine recognizers and deciders as follows:

- A TM T is a *recognizer* for a language L if ACCEPT(T) = L.
- A TM T is a *decider* for a language L if ACCEPT(T) = L and LOOP(T) = Φ. Equivalently, a TM T is a decider for L if ACCEPT(T) = L and REJECT(T) = $\overline{L}$.

We see that the above example M is a recognizer for the language $\texttt{a}\Sigma^*\texttt{a} \cup \texttt{b}\Sigma^*\texttt{b} \cup \Sigma$ but it is not a decider for this language.

Recall these definitions:

- A language L is *Turing-recognizable* if there is some TM that recognizes it.
- A language L is *decidable* if there is some TM that decides it.

We will show that there are languages that are: (1) Turing-recognizable but not decidable, and (2) languages that are not even Turing-recognizable. A language in the first category has the peculiar property that any TM that recognizes it must fail to terminate for some input strings. A language in the second category has the even more peculiar property that there is *no* TM that accepts exactly those strings belonging to that language. (By the Church-Turing thesis, this means that there is no *algorithm* that is able to return with an "accept" result for exactly those strings belonging to such a language.)