CSU390 Theory of Computation	Assignment #00
Spring 2006	January 10, 2006

Homework 00

Due: Tuesday, January 17, 2006

Note: This review assignment will be handled somewhat differently from all subsequent assignments. In particular:

- It will not be accepted late.
- You must attempt every problem.
- Every point you earn on this assignment will count toward your homework score as extra credit points.

1. a. (5 pts) For each of these, give the resulting set by listing out all its elements:
   - $\{1,2\} \cap \{1,3,5\}$
   - $\{1,2\} \cup \{1,3,5\}$
   - $\{1,2\} - \{1,3,5\}$
   - $\{1,3,5\} - \{1,2\}$
   - $\{1,2\} \times \{1,3,5\}$

   b. (5 pts) Given any set S, the power set of S, written $\mathcal{P}(S)$ or 2^S, is the set of all subsets of S. Write out $2^{\{1,2,3\}}$.

2. (10 pts) Suppose S is a finite set containing n elements, which we write as $|S| = n$. Use the product rule to derive a formula for $|2^S|$ in terms of n. (Hint: For each element of S, there are two possibilities: it is either in a particular subset or not.) The product rule says that if there are k_i options for selecting the i^{th} item and each item may be selected independently of all other items, there are $k_1 k_2 \ldots k_n$ ways of selecting a combination of all n items. Give a clear, concise mathematical argument justifying your answer.

There are essentially two ways we will define infinite sets in this class:

- using ellipses (i.e.,...); or
- using set-builder notation.

Here are two examples, defined using ellipses:

- $\mathcal{N}$ = the set of all natural numbers = $\{0, 1, 2, 3, \ldots\}$; and
- $\mathcal{Z}$ = the set of all integers = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Here is another example, which we define using both methods: The set of all natural numbers that are perfect squares is

$$\{0, 1, 4, 9, 16, 25, \ldots\} = \left\{n \mid n = m^2 \text{ for some } m \in \mathcal{N}\right\} = \left\{n^2 \mid n \in \mathcal{N}\right\}$$

3. (5 pts) Define the set of all odd natural numbers using both methods. (Your definition using set-builder notation should refer only to $\mathcal{N}$ and should not use properties like *odd* or *even*.)

4. Let $\mathcal{N}^{\text{odd}}$ denote the set of all odd natural numbers. A set S is said to be *closed* under an operation if the result of applying that operation to one or more elements of that set is always in the set. (How many elements the operation is applied to depends on how many operands that operation takes.) Prove or disprove:

   - (5 pts) $\mathcal{N}^{\text{odd}}$ is closed under addition.
   - (5 pts) $\mathcal{N}^{\text{odd}}$ is closed under multiplication.

5. (10 pts) Define an integer n to be even if $n = 2m$ for some integer m. Give a rigorous proof that, for any integer x, x is even if and only if $x + 2$ is even. Use only the definition just given, together with standard rules involving multiplication, addition, and subtraction of numbers. You will also need the fact that the set of integers is closed under these operations.

6. (10 pts) Give a rigorous proof that there is no natural number l such that $l \geq n$ for all $n \in \mathcal{N}$. I.e., prove that there is no largest natural number. You may use the fact that $n + 1 > n$ for any number n.

7. (10 pts) The nation of Automobilia is famous for having hundreds of automobile manufacturers. A government official of this nation proudly claims that at least one of their automobile manufacturers has provided, for each model they make with back seats, at least one cup holder within reach of one or more back-seat passengers.

   a. A disgruntled citizen of Automobilia asserts that this claim isn't true because he can point to one particular Automobilian automobile manufacturer, Umota, whose Zeta model has a back seat but no cup holders within reach of any back-seat passengers. Does this logic refute the claim? Explain clearly why or why not.

   b. Another disgruntled citizen asserts that this claim isn't true because she knows that every automobile manufacturer in Automobilia manufactures at least one model with no back seat. Does this logic refute the claim? Explain clearly why or why not.

   c. If neither of these arguments refutes the government official's claim, explain exactly what needs to be done to prove that the claim is false.