MATHEMATICS OF OPERATIONS RESEARCH
Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 0364-765X | EISSN 1526-5471 | 00 | 0000 | 0001

INFORMS
DOI 10.1287/xxxx.0000.0000

An Approximation Algorithm for the Bipartite Traveling Tournament Problem

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The Bipartite Traveling Tournament Problem (BTTP) is an NP-complete scheduling problem whose solution is a double round-robin inter-league tournament with minimum total travel distance. The 2n-team BTTP is a variant of the well-known Traveling Salesman Problem (TSP), albeit much harder as it involves the simultaneous coordination of 2n teams playing a sequence of home and away games under fixed constraints, rather than a single entity passing through the locations corresponding to the teams' home venues. As the BTTP requires a distance-optimal schedule linking venues in close proximity, we provide an approximation algorithm for the BTTP based on an approximate solution to the corresponding TSP.

We prove that our polynomial-time algorithm generates an 2n-team inter-league tournament schedule whose total distance is at most 1 + 2c/3 + (3-c)/3n times the total distance of the optimal BTTP solution, where c is the approximation factor of the TSP. In practice, the actual approximation factor is far better; we provide a specific example by generating a nearly-optimal inter-league tournament for the 30-team National Basketball Association (NBA), with total travel distance just 1.06 times the trivial theoretical lower bound.

Key words: scheduling; approximation algorithms; traveling tournament problem; traveling salesman
MSC2000 subject classification: Primary: 90B35; secondary: 90C59, 68W25
OR/MS subject classification: Primary: Networks/graphs: traveling salesman; secondary: distance algorithms; transportation: scheduling
History: Received May 25, 2012

1. Introduction Given a sports league with n teams, each with a home stadium, the Traveling Tournament Problem (TTP) asks for the optimal tournament schedule that minimizes the sum total of distances traveled by the n teams, subject to several conditions that ensure competitive balance. The TTP has direct applications to scheduling optimization, where professional sports leagues can make their regular season schedules more efficient, saving time and money, as well as reducing greenhouse gas emissions.

For any instance on n teams (with n even), the TTP requires the n team schedules to be properly coordinated so that every team plays a unique opponent in each time slot, with various constraints that limit the number of consecutive home and away games. Since its introduction [5], the TTP has emerged as a popular area of study within the operations research community [10] due to its complexity, where challenging benchmark problems remain unsolved. Research on the TTP has led to the development of powerful techniques in integer programming, constraint programming, as well as advanced heuristics such as simulated annealing [1] and hill-climbing [11]. Despite its apparent simplicity, the TTP is an NP-complete decision problem [15]. In fact, the TTP was only recently solved for a 10-team benchmark set [18] which has $n(n-1)=90$ total matches, and all benchmark sets with $n \geq 12$ remain unsolved [17].

In [6], we introduced the *Bipartite* Traveling Tournament Problem (BTTP), the inter-league extension of the TTP, and proved that it too is NP-complete. The $2n$-team BTTP consists of two n-team leagues X and Y, where every team plays one home game and one away game against each team in the other league. It appears that the BTTP is just as hard, if not harder than the TTP. For the BTTP, the best-known result is a 12-team benchmark set that was solved to optimality [7], which has $2n^2=72$ total matches. In other words, the TTP and BTTP are significantly harder than the well-known Traveling Salesman Problem (TSP), which despite its NP-hardness has been solved for an instance involving $85,900$ locations [2].

Given the NP-completeness of the BTTP, a natural question is to determine whether there exists a polynomial-time approximation to the optimal solution. In this paper, we answer that question by providing an approximation algorithm for the BTTP, motivated by recent papers that provided approximation algorithms for the general TTP [13, 20], as well as approximation algorithms for unconstrained TTP variants that considered different options for maximum tour length [9, 16, 19]. To do this, we connect the BTTP to the problem of finding a shortest Hamiltonian cycle (i.e., the TSP), as both problems ask for a distance-optimal schedule/route linking venues that are close to one another.

For any BTTP instance on $2n$ teams (with n teams in each league X and Y), let ψ be the total travel distance of the optimal solution. We wish to find an inter-league tournament schedule Γ whose total distance is no worse than $c^*\psi$. Let c be the approximation factor of the corresponding TSPs, where for each n-team league, the home venues represent the n locations in the salesman's tour. Given these approximate TSP solutions for the cities in X and Y, whose total distance is at most c times the length of the respective shortest Hamiltonian cycles, we create a BTTP solution Γ (i.e., a feasible inter-league tournament schedule) whose total distance is at most $c^*\psi$, where $c^*=1+\frac{2c}{3}+\frac{3-c}{3n}$.

Using the well-known Christofides algorithm [4] that guarantees a factor of $c=\frac{3}{2}$, our polynomial-time algorithm gives an approximation factor of $c^*=2+\frac{1}{2n}$. For the graphic metric, we apply a recent result [14] showing that $c=\frac{7}{5}$, which gives us an even better approximation ratio of $c^*=\frac{29}{15}+\frac{8}{45n}$. And for the Euclidean metric, we use the polynomial-time approximation scheme [3, 12] that guarantees a factor of $c=1+\epsilon$, which gives us $c^*\sim\frac{5}{3}+\frac{2}{3n}$.

The paper proceeds as follows. In Section 2, we formally define the $2n$-team Bipartite Traveling Tournament Problem. In Section 3, we describe our polynomial-time algorithm for constructing our desired schedule Γ. In Section 4, we prove that the total distance of Γ is at most $c^*=1+\frac{2c}{3}+\frac{3-c}{3n}$ times the optimal distance ψ. In Section 5, we demonstrate that our construction is actually much better in practice, and generate a nearly-optimal inter-league tournament schedule for the 30-team National Basketball Association (NBA). We show that our solution has total distance at most 1.06 times the optimal value, a far better ratio than the theoretical approximation of $c^*>\frac{5}{3}$. In Section 6, we conclude with several open problems and present ideas for further research.

2. The Bipartite Traveling Tournament Problem

Let there be $2n$ teams, with n teams in each league. Let X and Y be the two leagues, with $X=\{x_1,x_2,\ldots,x_n\}$ and $Y=\{y_1,y_2,\ldots,y_n\}$. Let D be the $2n\times 2n$ *distance matrix*, where entry $D_{p,q}$ is the distance between the home stadiums of teams p and q. By definition, $D_{p,q}=D_{q,p}$ for all $p,q\in X\cup Y$, and all diagonal entries $D_{p,p}$ are zero.

The BTTP requires a tournament lasting $2n$ days, where every team has exactly one game scheduled each day with no byes or days off. The objective is to minimize the total distance traveled by the $2n$ teams, subject to the following three conditions:

	1	2	3	4	5	6
x_1	y_1	y_2	y_3	$\mathbf{y_1}$	$\mathbf{y_2}$	$\mathbf{y_3}$
x_2	y_2	y_3	y_1	$\mathbf{y_2}$	$\mathbf{y_3}$	$\mathbf{y_1}$
x_3	y_3	y_1	y_2	$\mathbf{y_3}$	$\mathbf{y_1}$	$\mathbf{y_2}$
y_1	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	x_1	x_3	x_2
y_2	$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	x_2	x_1	x_3
y_3	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_1}$	x_3	x_2	x_1

	1	2	3	4	5	6
x_1	y_3	y_2	$\mathbf{y_1}$	$\mathbf{y_3}$	y_1	$\mathbf{y_2}$
x_2	$\mathbf{y_1}$	y_3	$\mathbf{y_2}$	y_1	y_2	$\mathbf{y_3}$
x_3	$\mathbf{y_2}$	y_1	$\mathbf{y_3}$	y_2	y_3	$\mathbf{y_1}$
y_1	x_2	$\mathbf{x_3}$	x_1	$\mathbf{x_2}$	$\mathbf{x_1}$	x_3
y_2	x_3	$\mathbf{x_1}$	x_2	$\mathbf{x_3}$	$\mathbf{x_2}$	x_1
y_3	$\mathbf{x_1}$	$\mathbf{x_2}$	x_3	x_1	$\mathbf{x_3}$	x_2

TABLE 1. Two feasible inter-league tournaments for $n=3$.

(a) *at-most-three*: No team may have a home stand or road trip lasting more than three games.
(b) *no-repeat*: A team cannot play against the same opponent in two consecutive games.
(c) *each-venue*: For all $1 \le i, j \le n$, teams x_i and y_j play twice, once in each other's home venue.

To illustrate, Table 1 provides two feasible tournaments satisfying all of the above conditions for the case $n=3$. In this table, as in all other schedules that will be subsequently presented, home games are marked in bold.

Following the convention of the TTP, whenever a team is scheduled for a road trip consisting of multiple away games, the team doesn't return to their home city but rather proceeds directly to their next away venue. Furthermore, we assume that every team begins the tournament at home, and returns home after playing their last away game. For example, in Table 1, team x_1 would travel a distance of $D_{x_1,y_1} + D_{y_1,y_2} + D_{y_2,y_3} + D_{y_3,x_1}$ when playing the schedule on the left and a distance of $D_{x_1,y_3} + D_{y_3,y_2} + D_{y_2,x_1} + D_{x_1,y_1} + D_{y_1,x_1}$ when playing the schedule on the right. The desired solution to the BTTP is the tournament schedule that minimizes the total distance traveled by all $2n$ teams subject to the given conditions.

Define a *trip* to be a pair of consecutive games not occurring in the same city, i.e., any situation where that team doesn't play at the same location in time slots s and $s+1$, and therefore has to travel from one venue to another. In Table 1, the schedule on the left has 24 total trips, while the schedule on the right has 32 trips. One may conjecture that the total distance of schedule S_1 is lower than the total distance of schedule S_2 iff S_1 has fewer trips than S_2.

To see that this is actually not the case, let the teams x_1, x_3, y_1, and y_2 be located at $(0,0)$ and let x_2 and y_3 be located at $(1,0)$. Then the schedule on the left has total distance 16 and the schedule on the right has total distance 12. So minimizing trips does not correlate to minimizing total travel distance; while the former is a trivial problem, the latter is extremely difficult, even for the case $n=3$.

Let BTTP* be the restriction of the BTTP to the set of tournament schedules where in any given time slot, the teams in each league either all play at home, or all play on the road. We can prove [6] that both BTTP and BTTP* are NP-complete.

The left schedule in Table 1 is a feasible solution of both the BTTP and BTTP*. We say that such schedules are *uniform*. While this uniformity constraint significantly reduces the number of potential tournaments, it allows us to quickly generate an approximate solution to the BTTP from an algorithm [8] based on minimum-weight 4-cycle-covers.

In the following section, we propose a different and more effective approach, using minimum-weight Hamiltonian cycles to generate an approximate solution to the BTTP. This will produce a c^*-approximation, where $c^* = 1 + \frac{2c}{3} + \frac{3-c}{3n}$ and c is the approximation factor of the TSP.

3. Polynomial-Time Construction Our work is inspired by several researchers who have obtained approximation algorithms for the Traveling Tournament Problem. In the standard TTP, no team can have a road trip or home stand longer than $k=3$ games; TTP variants allow for alternative values of k. For $k=3$, Miyashiro et. al. [13] found a $2 + \frac{9}{4(n-1)}$ approximation, which was later improved to $\frac{5}{3} + O(1/n)$ by Yamaguchi et. al. [20]. For $k=2$, Thielen and Westphal [16] found a $\frac{3}{2}+O(1/n)$ approximation. For a fixed $k>3$, Westphal and Noparlik [19] determined a 5.875 approximation independent of n and k. And finally, for arbitrary k (i.e., the unconstrained TTP), Imahori et. al. [9] obtained a 2.75 approximation independent of n.

We now present the first approximation algorithm for the Bipartite TTP. To do this, we let n be a multiple of 3. Thus, $n=3k$ for some integer $k\geq 1$. Let X and Y be the two leagues, with $X=\{x_1,x_2,\dots,x_{3k}\}$ and $Y=\{y_1,y_2,\dots,y_{3k}\}$. Let G be the graph with vertex set $X\cup Y$, where the weight of edge uv is the value of $D_{u,v}$ in the distance matrix, for all $u,v\in X\cup Y$.

Let H_X and H_Y respectively be the lengths of the shortest Hamiltonian cycles of the $3k$ cities in X and the $3k$ cities in Y. Letting c be the TSP approximation ratio, there is a polynomial-time algorithm to find Hamiltonian cycles with total distances at most cH_x and cH_y, respectively. Now re-label the vertices of X and Y so that these two (not-necessarily optimal) Hamiltonian cycles are $x_1x_2x_3\dots x_{3k}x_1$ and $y_1y_2y_3\dots y_{3k}y_1$.

For each $k\geq 1$, we build a *canonical* inter-league schedule, which we will call S_k. We first initialize the schedule for team x_1. For each $0\leq j\leq k-1$, set the following:

(a) Team x_1 plays team y_{3j+1} on day $6j+1$ on the road and on day $6j+4$ at home.
(b) Team x_1 plays team y_{3j+2} on day $6j+2$ on the road and on day $6j+5$ at home.
(c) Team x_1 plays team y_{3j+3} on day $6j+3$ on the road and on day $6j+6$ at home.

Thus, the schedule for x_1 is determined. Now define $f(i)=(i-1)+3\lfloor\frac{i-1}{3}\rfloor$ for each $1\leq i\leq 3k$, and set the schedule of team x_i to be the *cyclic shift* of team x_1's schedule, $f(i)$ units to the right. Specifically, if x_1 plays y_j on the road (at home) on day d^*, then x_i plays y_j on the road (at home) on day $d^*+f(i)$, where addition is calculated mod $6k$. Thus, we've described the schedule for each team in X, and so the schedule for each team in Y is now uniquely determined.

For example, if $k=3$ (and $n=9$), then we have $f(1)=0$, $f(2)=1$, $f(3)=2$, $f(4)=6$, $f(5)=7$, $f(6)=8$, $f(7)=12$, $f(8)=13$, $f(9)=14$. From this, we get Table 2, the inter-league tournament schedule S_3 for the teams in $X\cup Y$:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
x_1	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$
x_2	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$
x_3	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$
x_4	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$
x_5	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$
x_6	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$
x_7	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$
x_8	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$
x_9	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_7	y_8	y_9	$\boldsymbol{y_7}$	$\boldsymbol{y_8}$	$\boldsymbol{y_9}$	y_1	y_2	y_3	$\boldsymbol{y_1}$
y_1	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8	x_9
y_2	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8
y_3	x_8	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7
y_4	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6
y_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5
y_6	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4
y_7	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3
y_8	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2
y_9	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	x_7	x_8	x_9	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1

TABLE 2. The canonical schedule S_3, with $n=3k=9$ teams in each league.

By our application of cyclic shifts, we can quickly verify that each column of S_k consists of $6k$ unique elements; thus, every team plays a unique opponent in each time slot.

More formally, our $6k$-team inter-league tournament consists of $6k$ days of games, where team x_{3i+t} plays team y_{3j+u} on day $6(i+j)+(t+u-1)$ on the road, and on day $6(i+j)+(t+u+2)$

at home, for each 4-tuple (i,j,t,u), with $0 \le i \le k-1$, $0 \le j \le k-1$, $1 \le t \le 3$, and $1 \le u \le 3$. From this, it is easy to see that each S_k is feasible, as it satisfies the *at-most-three*, *no-repeat*, and *each-venue* conditions.

Note that the schedule's design is extremely efficient, as $4k$ of the teams take exactly $4k$ trips, the fewest number possible, while the other $2k$ teams (all in Y) take $4k+1$ trips, just one above the minimum.

We now calculate the total travel distance of S_k. Each trip can be represented by some edge uv in our graph G, with $u,v \in X \cup Y$.

For each team in $X \cup Y$, define a *hub edge* to be a trip that begins or ends in that team's home city, and a *non-hub edge* to be all other trips. By our construction of S_k, all non-hub edges for $x_i \in X$ are of the form $y_j y_{j+1}$, for some j (where index addition is taken mod $3k$). And similarly, all non-hub edges for $y_j \in Y$ are of the form $x_i x_{i+1}$ for some i. Furthermore, as S_k is an *inter-league* tournament, each hub edge will be of the form $x_i y_j$ for some $1 \le i,j \le 3k$.

Each of the $3k$ teams in X, as well as k teams in Y, have $2k$ hub edges and $2k$ non-hub edges. Furthermore, the remaining $2k$ teams in Y have $2k+2$ hub edges and $2k-1$ non-hub edges. Thus, the total travel distance of schedule S_k is the sum of the weights of $4k(2k)+2k(2k+2)=12k^2+4k$ hub edges and $4k(2k)+2k(2k-1)=12k^2-2k$ non-hub edges.

For each (i^*,j^*) with $0 \le i^*,j^* \le 3k-1$, define the tournament T_{i^*,j^*} as follows:

(a) Start with the canonical schedule S_k.

(b) Do a cyclic shift of $X=(x_1,x_2,\ldots,x_{3k})$, shifting all teams i^* places to the right.

(c) Do a cyclic shift of $Y=(y_1,y_2,\ldots,y_{3k})$, shifting all teams j^* places to the right.

In other words, the hub edge $x_i y_j$ in S_k corresponds to the hub edge $x_{i+i^*} y_{j+j^*}$ in T_{i^*,j^*}, with index addition mod $3k$. It is clear that for each pair (i^*,j^*), the tournament T_{i^*,j^*} also has $12k^2+4k$ hub edges and $12k^2-2k$ non-hub edges.

Finally, for each (i^*,j^*) with $0 \le i^*,j^* \le 3k-1$, define U_{i^*,j^*} to be the same tournament as T_{i^*,j^*}, but with X and Y exchanged. In other words, all occurrences of x_t are replaced by y_t, and vice-versa, for all $1 \le t \le 3k$. (So the hub edge $x_i y_j$ in one tournament becomes $x_j y_i$ in the other.) Clearly, each U_{i^*,j^*} is feasible, and also has $12k^2+4k$ hub edges and $12k^2-2k$ non-hub edges.

Now define the following:

$$\Delta = \sum_{i=1}^{3k}\sum_{j=1}^{3k} x_i y_j, \quad C_X = \sum_{i=1}^{3k} x_i x_{i+1}, \quad C_Y = \sum_{j=1}^{3k} y_j y_{j+1}.$$

Thus, Δ is the sum of the $9k^2$ hub edges, C_X is the total length of the cycle $x_1 x_2 \ldots x_{3k} x_1$, and C_Y is the total length of the cycle $y_1 y_2 \ldots y_{3k} y_1$. As explained earlier, we have labelled the vertices so that $C_X \le cH_X$ and $C_Y \le cH_Y$, where H_X and H_Y are the lengths of the shortest Hamiltonian cycles of X and Y.

There are $9k^2$ schedules of the form T_{i^*,j^*}, and $9k^2$ schedules of the form U_{i^*,j^*}. We now calculate V, the *average* total travel distance of the $18k^2$ schedules.

Due to our cyclic shifts, each of the $9k^2$ hub edges is equally likely to occur. Similarly, each of the $6k$ non-hub edges is equally likely to occur. Thus,

$$V = \frac{12k^2+4k}{9k^2}\Delta + \frac{12k^2-2k}{6k}(C_X+C_Y) \le \left(\frac{4}{3}+\frac{4}{9k}\right)\Delta + \left(2ck - \frac{c}{3}\right)(H_X+H_Y).$$

We now verify the following lemma.

Lemma 1. *Given an instance of $2n$ teams (where n is a multiple of 3), with n teams in X and n teams in Y, there exists a polynomial-time algorithm to find a feasible inter-league tournament schedule Γ with total distance at most V.*

As n is a multiple of 3, let $n = 3k$. The canonical schedule S_k can be constructed in $O(k^2)$ time, and the total travel distance of S_k can also be determined in $O(k^2)$ time. Similarly, it requires $O(k^2)$ time to calculate the total travel distance for each of the $18k^2$ schedules in T_{i^*,j^*} and U_{i^*,j^*}, by applying the appropriate cyclic shifts and variable exchanges. This requires a total of $O(k^4)$ computations, which is equivalent to $O(n^4)$ because $n = 3k$.

Among all the $18k^2$ possible options, select Γ as the inter-league tournament schedule with the minimum possible distance. Since the average of these $18k^2$ total distances is V, the total travel distance of Γ cannot exceed V. Therefore, we have found the desired schedule Γ, in $O(n^4)$ time. Q.E.D.

4. Determining the Approximation Ratio Given an instance on $2n$ teams, let ψ be the total travel distance of the optimal inter-league tournament between the n teams in X and the n teams in Y.

For each $1 \leq i \leq 3k$, let ILB_{x_i} to be the *independent lower bound* for team x_i, the minimum possible distance that can be traveled by x_i in order to complete its games, independent of the other teams' schedules. Similarly define ILB_{y_j} for each $1 \leq j \leq 3k$. Then we trivially have

$$\psi \geq \sum_{i=1}^{3k} ILB_{x_i} + \sum_{j=1}^{3k} ILB_{y_j}.$$

Each team's minimum-distance schedule can be graphically represented as a union of cycles of length at most 4, where each cycle has a common root vertex. Figure 1 illustrates this concept. Note that the ILB is simply the sum of all the edge weights.

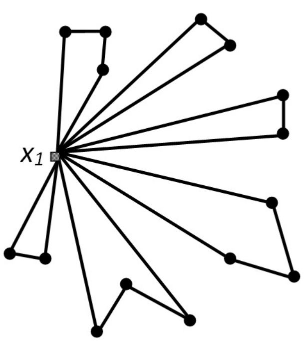

FIGURE 1. A possible schedule for team x_1.

We prove the following lemma.

LEMMA 2. *For any instance on $2n$ teams, with $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, we have*

$$ILB_{x_i} \geq \frac{2}{3}\sum_{j=1}^{3k} x_i y_j, \quad ILB_{y_j} \geq \frac{2}{3}\sum_{i=1}^{3k} x_i y_j, \quad ILB_{x_i} \geq H_Y, \quad ILB_{y_j} \geq H_X.$$

As illustrated in Figure 2 (left), consider one of the 4-cycles rooted at vertex x_1. In playing this 3-game road trip against the teams in $\{y_1, y_2, y_3\}$, team x_1 travels a total distance of $D_{x_1,y_1} + D_{y_1,y_2} + D_{y_2,y_3} + D_{y_3,x_1}$. By the Triangle Inequality, we have

$$\begin{aligned}
3(D_{x_1,y_1} + D_{y_1,y_2} + D_{y_2,y_3} + D_{y_3,x_1}) &\geq 3D_{x_1,y_1} + D_{y_1,y_2} + D_{y_2,y_3} + 3D_{y_3,x_1} \\
&= 2D_{x_1,y_1} + (D_{x_1,y_1} + D_{y_1,y_2}) + (D_{y_2,y_3} + D_{y_3,x_1}) + 2D_{y_3,x_1} \\
&\geq 2D_{x_1,y_1} + D_{x_1,y_2} + D_{y_2,x_1} + 2D_{y_3,x_1} \\
&= 2(D_{x_1,y_1} + D_{x_1,y_2} + D_{x_1,y_3}).
\end{aligned}$$

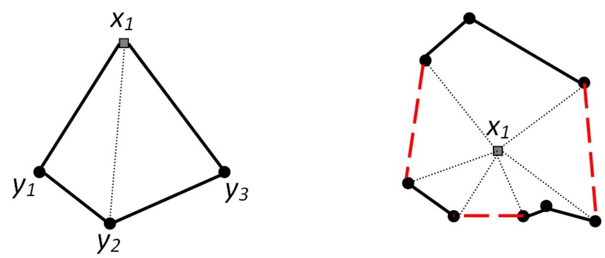

Figure 2. A visual illustration of Lemma 2.

Therefore, in this 3-game road trip, the travel distance of x_1 is at least $\frac{2}{3}\sum_{j=1}^{3} x_i y_j$. Clearly, this inequality is satisfied for any road trips of length less than three: for example, in the 2-game road trip $\{y_4, y_5\}$, the travel distance is at least $\sum_{j=4}^{5} x_i y_j \geq \frac{2}{3}\sum_{j=4}^{5} x_i y_j$. Adding up these inequalities for each of the cycles rooted at x_i, we conclude that $ILB_{x_i} \geq \frac{2}{3}\sum_{j=1}^{3k} x_i y_j$. By symmetry, we have $ILB_{y_j} \geq \frac{2}{3}\sum_{i=1}^{3k} x_i y_j$.

Suppose team x_1 plays their games against $y_1, y_2, \ldots, y_n$, in that order. Then as illustrated in Figure 2 (right), the total travel distance of x_1 is at least the distance of the Hamiltonian cycle $y_1 y_2 \ldots y_n y_1$, by the Triangle Inequality. Since H_Y is the length of the shortest Hamiltonian cycle (i.e., the TSP-solution for the n-city instance Y), we have $ILB_{x_i} \geq H_Y$. And by symmetry, we also have $ILB_{y_j} \geq H_X$. Q.E.D.

We can now prove the main result of our paper.

Theorem 1. *Given an instance of $2n$ teams (where n is a multiple of 3), with n teams in X and n teams in Y, there exists a polynomial-time algorithm to find a feasible inter-league tournament schedule Γ with total distance at most $c^*\psi$, where $c^* = 1 + \frac{2c}{3} + \frac{3-c}{3n}$, ψ is the total distance of the optimal solution, and c is the approximation ratio of the corresponding TSPs.*

By Lemma 1, there is a polynomial-time algorithm that constructs a feasible inter-league schedule Γ with total distance Φ, where

$$\Phi \leq V \leq \left(\frac{4}{3} + \frac{4}{9k}\right)\Delta + \left(2ck - \frac{c}{3}\right)(H_X + H_Y).$$

We now prove that this schedule Γ satisfies $\Phi \leq c^*\psi$, which will complete our proof.

By Lemma 2, we have the following two inequalities:

$$\psi \geq \sum_{i=1}^{3k} ILB_{x_i} + \sum_{j=1}^{3k} ILB_{y_j} \geq \frac{2}{3}\sum_{i=1}^{3k}\sum_{j=1}^{3k} x_i y_j + \frac{2}{3}\sum_{j=1}^{3k}\sum_{i=1}^{3k} x_i y_j = \frac{4}{3}\Delta,$$

$$\psi \geq \sum_{i=1}^{3k} ILB_{x_i} + \sum_{j=1}^{3k} ILB_{y_j} \geq \sum_{i=1}^{3k} H_Y + \sum_{j=1}^{3k} H_X = 3k(H_X + H_Y).$$

Recalling that $n = 3k$ and $c^* = 1 + \frac{2c}{3} + \frac{3-c}{3n}$, we conclude that

$$\frac{\Phi}{\psi} \leq \left(\frac{4}{3} + \frac{4}{9k}\right)\frac{\Delta}{\psi} + \left(2ck - \frac{c}{3}\right)\frac{(H_X + H_Y)}{\psi} \leq \left(\frac{4}{3} + \frac{4}{9k}\right)\frac{3}{4} + \left(2ck - \frac{c}{3}\right)\frac{1}{3k} = c^*.$$

Q.E.D.

5. Application Therefore, we have established our main result, the existence of a polynomial-time algorithm that constructs an inter-league schedule Γ whose total distance Φ is at most $c^* = 1 + \frac{2c}{3} + \frac{3-c}{3n}$ times the distance of the optimal solution.

In practice, Γ is much better than a c^*-approximation, as it takes the minimum-distance schedule among the set of $18k^2$ schedules considered by our algorithm, and often this *minimum* distance Φ is far less than the *average* distance V that was used to derive our approximation ratio.

To illustrate this principle, consider the unit square $PQRS$ with a "taxicab-distance" metric, where the distance between any two points is the sum of the differences between corresponding coordinates. Specifically, suppose that $P = (0,0)$, $Q = (0,1)$, $R = (1,1)$, and $S = (1,0)$. Then $D_{PQ} = D_{QR} = D_{RS} = D_{SP} = 1$ and $D_{PR} = D_{QS} = 2$.

Consider an inter-league tournament with $n = 6$, with the six teams of X located at P and R (with three at each point), and the six teams of Y located at Q and S (with three at each point). Then clearly the shortest Traveling Salesman tour for X is $H_X = 2D_{PR} = 4$. Similarly, $H_Y = 2D_{QS} = 4$. Thus, we label the teams so that $\{x_1, x_2, x_3\}$ are at P, $\{x_4, x_5, x_6\}$ are at R, $\{y_1, y_2, y_3\}$ are at Q, and $\{y_4, y_5, y_6\}$ are at S. We can readily verify that $\Delta = \sum\sum x_i y_j = 36$.

Since we can solve the 6-city TSPs for both X and Y, we have $c = 1$, $n = 6$, and $c^* = 1 + \frac{2}{3} + \frac{2}{18} = \frac{16}{9}$. To generate this $\frac{16}{9}$-approximation, first we create the canonical schedule S_2, illustrated in Table 3 below.

	1	2	3	4	5	6	7	8	9	10	11	12
x_1	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$
x_2	$\boldsymbol{y_6}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$
x_3	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$
x_4	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$
x_5	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$
x_6	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_4}$	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_1	y_2	y_3	$\boldsymbol{y_1}$
y_1	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6
y_2	x_6	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5
y_3	x_5	x_6	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4
y_4	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2	x_3
y_5	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1	x_2
y_6	x_2	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	x_4	x_5	x_6	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1

TABLE 3. The canonical schedule S_2, with $n = 3k = 6$ teams in each league.

As explained in Section 3, there are $9k^2 = 36$ schedules of the form T_{i^*,j^*}, and $9k^2 = 36$ schedules of the form U_{i^*,j^*}. Thus, the average total distance of these 72 schedules is

$$V = \left(\frac{4}{3} + \frac{4}{9 \cdot 2}\right)\Delta + \left(2 \cdot 2 - \frac{1}{3}\right)(H_X + H_Y) = \frac{14}{9} \cdot 36 + \frac{11}{3} \cdot 8 = \frac{256}{3}.$$

Since each team's independent lower bound is 4, we have $\psi \geq 12 \times 4 = 48$. It is straightforward to find a feasible schedule obtaining this optimal value. Thus, $\psi = 48$, giving us the desired ratio of $\frac{256}{3 \cdot 48} = \frac{16}{9}$. However, we can do far better than an approximation ratio of $\frac{16}{9}$, by selecting the best schedule Γ from our 72 possible options. The optimal choice is the canonical schedule S_2 (corresponding to $T_{0,0}$), which has total distance $\Phi = 56$. Thus, we have a ratio of $\frac{\Phi}{\psi} = \frac{56}{48} = \frac{7}{6}$. So in this simple example, the average approximation ratio is $\frac{16}{9}$, but in fact our algorithm produces a $\frac{7}{6}$-approximation in polynomial time.

For a harder example, consider the BTTP for the 30-team National Basketball Association (NBA), a problem first presented in [6]. We begin by obtaining TSP solutions for both the 15-team Western Conference and the 15-team Eastern Conference, which we do by "eyeballing" the map

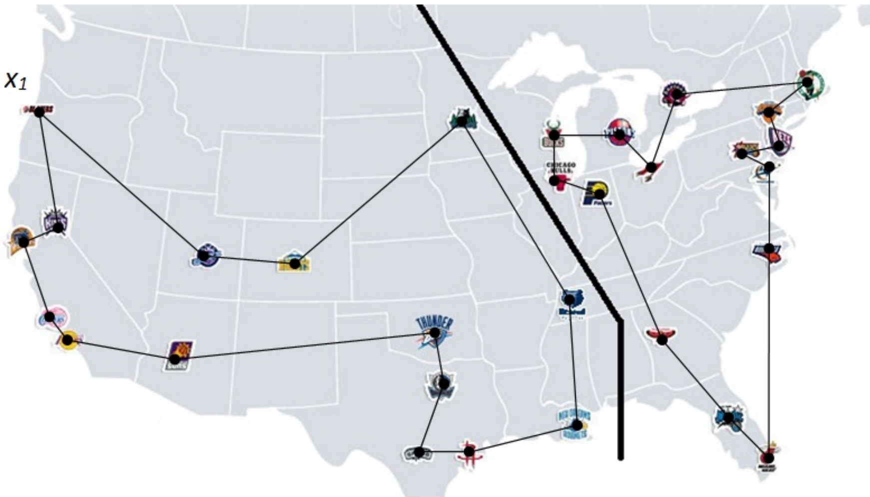

FIGURE 3. Optimal TSP tours for the NBA Western and Eastern Conferences.

and connecting cities that appear closest to each other. These two Hamiltonian cycles, seen in Figure 3, are indeed optimal, as verified by an exhaustive computer search.

Using the 30×30 distance matrix [8], we can show that $\Delta = 338618$, $H_X = 5691$, and $H_Y = 3603$ (all distances in miles). As the distances satisfy the Euclidean metric, we have $c = 1$. Since $n = 15$, we have $c^* = 1 + \frac{2}{3} + \frac{2}{45} = \frac{77}{45} \sim 1.71$.

In the proof of Theorem 1, we used the inequalities $ILB_{x_i} \geq \frac{2}{3}\sum_{j=1}^{3k} x_i y_j$ and $ILB_{x_i} \geq H_Y$ to derive our approximation ratio c^*. The latter inequality is a terrible estimate for the NBA instance, as the majority of cities in the Western Conference are so far apart from the cities in the Eastern Conference, which makes the values of ILB_{x_i} significantly larger than that of H_Y. For example, the Portland Trailblazers (labelled as x_1 in Figure 3 above) must make at least five trips out east to play its 15 inter-league road games. A computer search shows that $ILB_{x_1} = 23610$, which is just 8% higher than $\frac{2}{3}\sum_{j=1}^{3k} x_i y_j = 21844$, but 555% higher than $H_Y = 3603$.

So in calculating our lower bound ψ, we only use the inequalities $ILB_{x_i} \geq \frac{2}{3}\sum_{j=1}^{3k} x_i y_j$ and $ILB_{y_i} \geq \frac{2}{3}\sum_{i=1}^{3k} x_i y_j$, which implies $\psi \geq \frac{4}{3}\Delta \sim 451491$.

Since there are $n = 15$ teams in each league, we have $k = 5$, and so we use our canonical schedule S_5 to build an inter-league schedule with 30 total teams. By our earlier analysis, we can calculate the average total distance of the $18k^2 = 450$ schedules of the form T_{i^*,j^*} and U_{i^*,j^*}:

$$V = \left(\frac{4}{3} + \frac{4}{9 \cdot 5}\right)\Delta + \left(2 \cdot 5 - \frac{1}{3}\right)(H_X + H_Y) = \frac{64}{45} \cdot 338618 + \frac{29}{3} \cdot 9294 \sim 571422.$$

And therefore, we have the approximation ratio $\frac{\Phi}{\psi} \leq \frac{571432}{451491} \sim 1.26$. However, we can do even better by selecting the minimum-distance schedule among all 450 possibilities, whose total distances range from 549467 to 597088. Taking the schedule Γ with total distance $\Phi = 549467$, we have $\frac{\Phi}{\psi} \leq \frac{549467}{451491} \sim 1.22$.

As $n = 15$ is a relatively small case, we can quickly determine each ILB_{x_i} and ILB_{y_j} using a brute-force enumeration. From this, we can prove that $\sum_{i=1}^{15} ILB_{x_i} + \sum_{j=1}^{15} ILB_{y_j} = 517932$, which gives us the improved lower bound $\psi \geq 517932$. Therefore, our approximation ratio can be improved even further, to $\frac{\Phi}{\psi} \leq \frac{549467}{517932} \sim 1.06$.

Hence, we have found a 1.06-approximation to the 30-team NBA inter-league problem, which is significantly better than the conservative $c^* \sim 1.71$-approximation guaranteed by our polynomial-time algorithm.

6. Conclusion We conclude the paper with three problems for further research.

In this paper, we determined a polynomial-time c^*-approximation for the Bipartite Traveling Tournament Problem (BTTP), where $c^* = 1 + \frac{2c}{3} + \frac{3-c}{3n}$. Our construction only works for the case $n \equiv 0 \pmod 3$, and so a natural question is whether our method can also be applied to the cases $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$.

We also ask whether the approximation ratio $1 + \frac{2c}{3} + O(\frac{1}{n})$ is best possible. We suspect that we can do better, by tightening our inequalities to produce a better ratio, as we did with the 30-team NBA instance. We also wonder whether we can derive a better approximation algorithm by considering minimum-weight triangle packings instead of minimum-weight Hamiltonian cycles in the construction of our canonical schedule S_k. It is clear that Hamiltonian cycles, the approach taken in this paper, provides a good approximation algorithm. However, other methods might produce even better results.

Finally, we can analyze variants of the BTTP that modify k, the maximum length of a road trip or home stand. Just as various researchers [9, 16, 19] have analyzed these variants for the *intra*-league TTP, it would be interesting to derive analogous results for the *inter*-league BTTP, beyond the $k = 3$ case considered in this paper.

Acknowledgments. This research has been partially supported by the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research), the C & C Foundation, the Kayamori Foundation, and the Inoue Research Award for Young Scientists.

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