PROBLEM SOLVING: DEFINITION, ROLE, AND PEDAGOGY

RÉSOLUTION DE PROBLÈMES : DÉFINITION, RÔLE, ET PÉDAGOGIE ASSOCIÉE

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INTRODUCTION

In this working group, we engaged in three main activities:

- We discussed and attempted to come to a definition of a 'good' problem (specifically in educational contexts), and a definition of 'problem solving'.
- We discussed the role of problem solving in K-12 classrooms, especially problem solving as a method of teaching rather than a specific topic to teach in the curriculum.
- We discussed how problem solving should be addressed with future educators.

Dans ce groupe de travail, nous avons discuté de trois thèmes principaux :

- *Les définitions d'un « bon » problème, et de la « résolution de problèmes ».*
- *Les rôles de l'activité de résolution de problèmes dans les écoles de la maternelle à la douzième année, d'autant plus que la résolution de problèmes n'est plus seulement un objet d'enseignement, mais aussi un outil didactique pour enseigner d'autres sujets particuliers du programme.*
- *Les approches pour aborder la résolution de problèmes avec les futurs enseignants.*

DAY ONE

Here was our focus for the first day: *Discuss and attempt to come to a definition of a ‘good’ problem for learning, and a definition of ‘problem-solving’*.

We started with this warm-up question:

> Each of the sixteen squares has a ‘point value’, as indicated. Your goal is to collect as many points as possible, starting at the top-left corner and ending at the bottom-right corner. You may pass through each room at most once. What is the maximum number of points that you can collect?

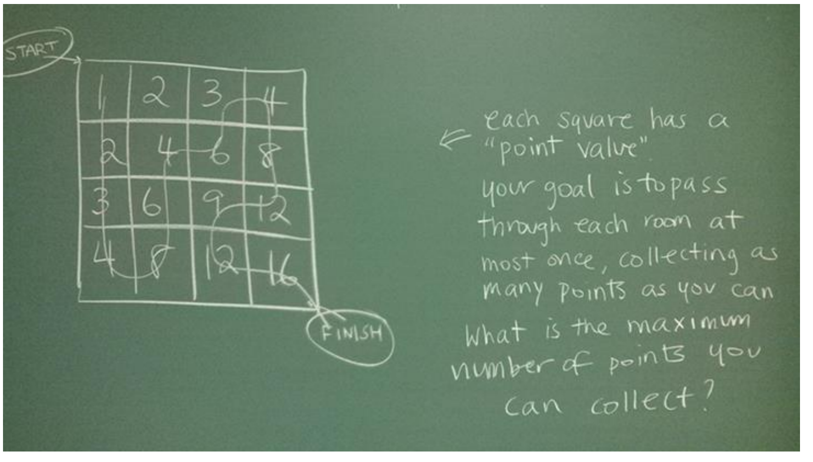

Figure 1

Participants shared various reasons why this is a good ‘problem’:

- it is easy to start and it is easy to make more difficult;
- it does not require algebraic tricks;
- chessboard colouring provides a nice anchor to existing knowledge;
- the grid has ‘distractors’ in that the sixteen numbers are irrelevant; and
- one can quickly come up with a near-optimal answer, but needs to do some work to obtain the optimal answer (of 98 points).

However, participants also mentioned a weakness of this problem: the chessboard colouring is heuristically non-transparent, i.e., how would someone come up with that approach? Of course there are ways to address this in a teaching context, but this need for a ‘trick’ or an ‘aha! insight’ makes using this problem less straightforward.

This notion of making progress is important to the teaching of mathematical problem solving. We discussed the need to present problems that get the students hooked, where they can make incremental progress and see success; this is true whether we are teaching in elementary school or high school, teaching undergraduates, or conducting pre-service or in-service work for educators.

A good problem should not rely on a single ‘aha’ insight that renders the problem trivial, where it is impossible to make progress without seeing the trick. In this light, the question above is not a good ‘problem’, in spite of its many other strengths. (For discussion of this issue in relation to proving, see Reid & Tanguay, 2012).

The problem mentioned above can be good for something, for example to introduce the applications of grid colouring, but not so good for something else, for example to convince the students that they can make progress by themselves in solving any problem. In order to use problems in class efficiently, we need to decide for each problem for what learning it is good or bad.

SHARING PROBLEMS

We split into groups of four, in which we shared our favourite problems and selected one or two to offer to the others. We posted the problems on the walls, and milled about, working on whatever problems caught our interests. The problems presented included the following:

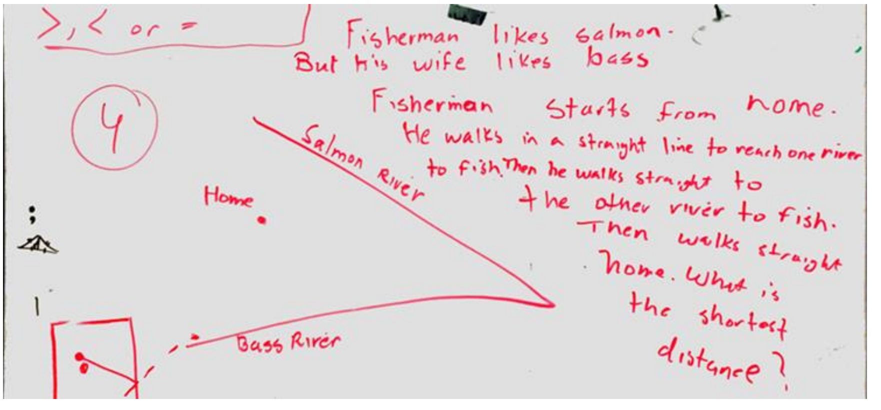

Figure 2. The Fisherman Problem.

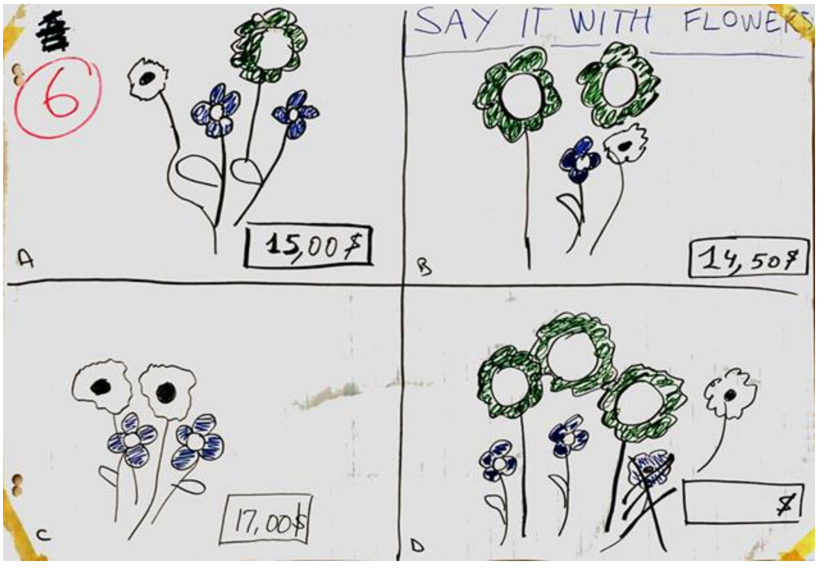

Figure 3. The Flower Problem.

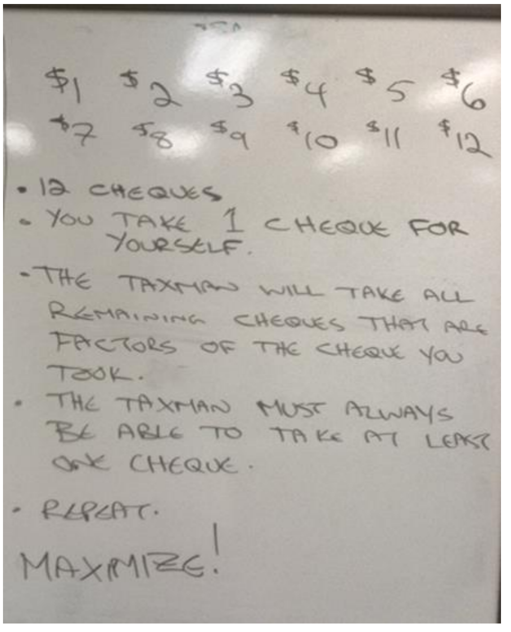

Figure 4. The Taxman Problem.

Inspired by these problems, we discussed characteristics of what makes them 'good'.

- A good problem is often one where an insight leads to further mathematics, as in the Fisherman Problem (Figure 2). The Fisherman Problem can be generalised or simplified in several ways that develop mathematical understanding.
- A good problem is often accessible and non-threatening, as in the Flower Problem (Figure 3) which was presented using a non-verbal context. This avoids the issue of having to understand the words of a problem, before being able to engage with the problem itself. As someone noted, the problem with word problems is the words. This also reminded us that what makes a problem a problem is dependent on the problem solver. For us, one possible solution path, using algebra, was obvious, and hence the problem was not a problem. We could make it one, however, by imposing constraints, such as not using algebra, or operating on the sets of flowers concretely rather than abstracting them symbolically. Following this logic, the interpretation of a task (word problem or picture problem) in a mathematical way can be problematic for someone. So the initial mathematical interpretation of a task is a cornerstone for problem solving.
- A good problem is often generalizable, as in the Taxman Problem (Figure 4) where we can replace 12 cheques with 18 cheques or with N cheques. And this problem is accessible to Grade 4 children while also inspiring questions that would challenge upper-year computer science majors. But we need to be aware of what each of these groups can learn through the process of solving.
- A good problem is often grounded in a real-life context (e.g. maximizing a profit), which engages students and enables them to connect the topics they learn in class to the issues they care about.

Peter introduced the term 'unicorn problems' to describe those rare problems that are not only good for student learning, but are *immediately engaging in almost any context with almost any population*. Peter offered the Taxman Problem (Figure 4) as an example of a unicorn problem.

On the other hand, the Island-Moat Problem (Burger & Starbird, 2005) below rests on a single 'aha' insight (or trick) of how to lay down the two planks to reach the island (Figure 5).

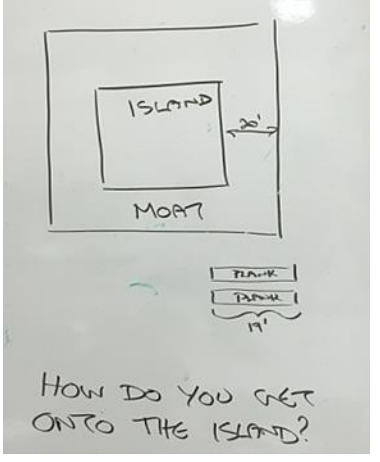

Figure 5. Island-Moat Problem (Burger & Starbird, 2005).

But we should note that in the original formulation, the Island-Moat Problem is presented orally or textually, rather than visually. Thus, there is some work for students to do in order to move from one form of mathematical representation to another; once the above visual representation is attained, a clever insight is required to solve the problem. It is also possible that the use of physical objects, planks of appropriate size, can facilitate the clever insight.

This problem can be improved, and our Working Group spent some time discussing the different ways to do that: for example, determine the shortest possible length of the two planks for which you can get onto the island.

The problems we discussed also reminded us of some definitions that have been offered by others:

> *First, a problem is only a* Problem *(as mathematicians use the term) if you don't know how to go about solving it. A problem that holds no 'surprises' in store, and that can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an* exercise*. This latter description applies to most of the 'word problems' that students encounter in elementary school, to 'mixture problems', 'rate problems', or other standard parts of the secondary curriculum. Dealing with them is certainly an important part of learning mathematics, but (unless the context is unusual) working such exercises is not generally considered 'doing problem solving'.*
>
> *Second, a problem is not a* Problem *until one wants to solve it. (The presumption in most problem books is that the reader does. Why else would he or she be looking at the problem book?) Once one wants to solve a* Problem*, there is an emotional and intellectual commitment to the solution, and the risks and rewards concomitant with that commitment.* (Schoenfeld, 1983, p. 41)

A good problem should not be procedural: if it is routine for us, then it is an exercise and not a problem. We need to get 'stuck', to find an obstacle that we have to overcome, in order for a question to become a problem. The theory of *Zone of Proximal Development* proposed by Vygotsky (1991) explains, "The teaching oriented toward already completed cycles of development happens to be vain from the point of view of the child's general development, it does not lead the development but follows the tail of it" (p. 386). Thus, a problem needs to present some difficulty to the learner in order to serve the purpose of knowledge development.

In our discussions, we reminded each other that a 'problem' is relative to each individual, based on their culture, experience, confidence, and background. What is a problem for me might not be a problem for you. And what was a problem for you one year ago might not be a problem for you today. However, this way to define a problem cannot be used to describe a mathematical task per se, because it requires the presence of a solver. Thus, the terminology of 'problem' is not very practical. Many researchers in the field use 'task' terminology, which is associated with the formulation of the task, and not in relation to a particular solver. A task can become a 'good problem' for some solvers in some conditions and a routine problem (or not a problem) for other solvers or in different conditions.

This connects to Brousseau's notion of *problem situations*. For Brousseau (1971), problems are tasks. The teacher needs to do something special, to create a milieu, for the task to let students develop a particular knowledge.

> *L'élève acquiert ces connaissances par diverses formes d'adaptation aux contraintes de son environnement. En situation scolaire, l'enseignant organise et constitue un milieu, par exemple un problème, qui révèle plus ou moins clairement son intention d'enseigner un certain savoir à l'élève mais qui dissimule suffisamment ce savoir et la réponse attendue pour que l'élève ne puisse les obtenir que par une adaptation personnelle au problème proposé. La valeur des connaissances acquises ainsi dépend de la qualité du milieu comme instigateur d'un fonctionnement « réel », culturel du savoir, donc du degré de refoulement a-didactique obtenu.* (Brousseau, 1971, p. 40)

Participants noted the following:

1. A mathematical problem is a situation where a solution is needed and is not obvious (requiring an exploration, a cognitive obstacle, motivational) for a person (intention, culture) according to *a priori* analysis, can/should potentially involve mathematics at some time for a specific part of it, and probably invent some.
2. Solve a problem: recognize the 'problem', use our previous knowledge and understanding of the problem, to find a solution that satisfies us.
3. A mathematical problem is what at least one person considers as such, many people choose to engage and find it meaningful.
4. A mathematical problem represents connection between mathematics and real life.
5. A mathematical problem is a question requiring a mathematical read and for which the solution does not exist.

We ended Day 1 by having each participant define 'problem' and 'problem-solving' on their own, with each participant submitting their definition on a sheet of paper. Based on the collected input, here is our first draft at our collective group's definition of these two key terms:

- A mathematical problem is a clearly-stated question whose aim is to challenge the one attempting the question by forcing them to resolve a cognitive obstacle in order to develop mathematical reasoning and maturity. A good problem requires challenge, is ideally accessible and engaging to people of most/many levels, and necessitates that the problem-solver get stuck or challenged. Often the key insight is found by making a connection between an abstract concept and a real-life context, or finding a connection between two or more fields of mathematics. A mathematical question is not a 'problem' for everyone, and is relative to an individual's prior experiences with the subject and to his/her personal characteristics as a learner.
- Mathematical problem solving is a cognitive activity that aims to resolve a challenging question by getting 'unstuck'. It is rarely a routine process (step 1 do this, step 2 do that), and instead is a non-linear process that necessitates conceptual understanding rather than routine calculations, and requires us to try different strategies and methods to discover key insights and make progress. Through our

struggle and perseverance, sometimes we can find multiple solutions to the problem, requiring different insights and techniques that relate to our previous experiences with mathematics. An unexpected benefit is that solving these problems deepens our passion for the subject.

DAY TWO

Here was our focus for the second day: *Discuss the role of problem-solving in K-12 classrooms, especially problem solving as a method of teaching rather than a specific topic to teach in the curriculum.*

We started with a two-minute video showing a Grade 2 student solving the following task, where she got the wrong answer. The students were asked first to read the problem and then propose a strategy to solve it:

> I have 13 tokens. I hide 7 of them in my left hand and the others in my right hand. How many tokens do I have in my right hand?

In the video, the student drew 13 circles, then 7 other circles. She said that 7 are in the left hand, so the others (13 on her picture) are in the right hand.

We asked ourselves, "*Is this a real problem?*" It seems that the student had the preconceived expectation of what to do: she immediately started drawing circles. Some of our students have preconceived expectations (what Brousseau (1971) would call a « *contrat didactique* »), where they want to apply the exact recipe they learned in class to solve each question.

For the Grade 2 child, she used a procedure learned in class—drawing circles—which she could not control because of a lack of conceptual understanding of the task. What was problematic to the student was not to calculate the answer to 13 – 7, but to correctly interpret the described story in a mathematical way.

A mathematical interpretation of a situation, task, or problem, which is connecting a real-life context with mathematical concepts, is the first step to the solution and it is the first challenge elementary students have in problem solving.

Educators use problem solving to stimulate learning, to ask questions that are *complex*, rather than *complicated*. In a good problem, words are not a barrier where we need to read pages to understand what a problem is saying. Language is often a barrier, especially for students who do not speak the official school language well. However, we must ask ourselves whether the words pose a 'barrier' or pose a 'challenge'. The answer depends on the learner's language abilities and on the purpose of the solving. According to Vygotsky (1934), language is one of the main reasoning tools, thus it should be developed together with other ways of mathematical reasoning.

Once we understand what the problem is saying, we need to understand it mathematically. This is a large part of the challenge for a learner: once this understanding has taken place, much of the problem has been solved. A problem can be presented as a word story, as a picture (flower problem) or another way (equation). Sometimes the problem can be solved the way it is presented (solve the equation) or transformed into another representation (flowers into equations; wording into schema).

In these transformations, the mathematical meaning should be preserved. Different representations require different parts of the brain to be involved to understand them, where wording is decoded mainly by the left hemisphere and pictures by the right hemisphere. If we can pose problems and solve problems that require us to make connections between the left brain and the right brain, growth occurs (Wachsmuth, 1981). We saw another video, showing the "cinnamon heart" problem, where the students analyzed the following text:

> Rene has 11 cinnamon hearts. He ate 6 of them. There remain 8 cinnamon hearts.

The students realized that there was an error in the text. They made the assumption that the error is at the end, that the number 8 is wrong. However, the error could have been at the beginning or in the middle, i.e., the 11 or 6 could have been wrong. The teacher asked the students to represent the situation in order to mathematically analyze it. The task was quite challenging for students, and many of them were not successful in constructing a representation.

This task satisfies many possible definitions of a problem (1, 2, 3, and 5 in the above list). However, it does not present a connection with real-life but rather a disconnection. This disconnection creates an element of surprise and a need for further investigation (Schoenfeld, 1987). It serves to engage students in a profound mathematical analysis and allow for mathematical knowledge and development.

In this mathematical task, the class analyzed the problem, rather than just solving it. This led to an important discussion about our role as educators: Should we make a clear distinction between problem analysis/modeling and problem solving, or rather see the problem analysis/modeling as an important part of problem solving?

We said that a 'conventional' problem can become a great learning opportunity by pushing it: by putting a small twist on it. So much problem solving occurs when we as teachers have the confidence to encourage open-ended problem solving, such as giving them Pascal's Triangle and asking them to find as many patterns as they can.

Let's give our students great problem-solving experiences, which require us to be open to being surprised, having to go off script from our lesson plan. We can include non-conventional 'problem-solving' opportunities in the curriculum which can help develop the reasoning required for problem solving.

At this point, we analyzed three of the problems from Day 1, asking what they are good for, and for whom they are good. In other words, we did a didactic analysis, thinking about how we would present these problems in a lesson.

1. The Fisherman Problem (Figure 2): we presented this problem as a sequence of activities, suitable for a Grade 12 class or for an undergraduate problem-solving class (see Figure 6). In this carefully structured sequence of activities, students got stuck and had to learn how to get unstuck. The 'reflection' insight is an example of epistemological justification, which motivates to the student why we need to learn the technique (not what it is or how it works).
2. The Flower Problem (Figure 3): this problem was applicable to Grade 2 and to Grade 9 students, where they develop new knowledge and practice existing knowledge. One could solve this by bringing actual flowers, so that they can solve the problem by manipulating flowers and building the bouquet themselves. Elena had difficulty associating white flowers in two bouquets because they were drawn slightly differently. The visual representation (probably any particular representation) of a problem can be an obstacle/challenge, as well as wording in a natural language.

3. The Taxman Problem (Figure 4): this problem can be used in any grade level, from Grade 4 students to upper-year undergraduates. This is a playful game that is easy to understand, students can pay the taxman as many times as they need, and easy to make difficult by adding more bills. It is resistant to poor delivery (representation is simple), and minimal teacher input is required.

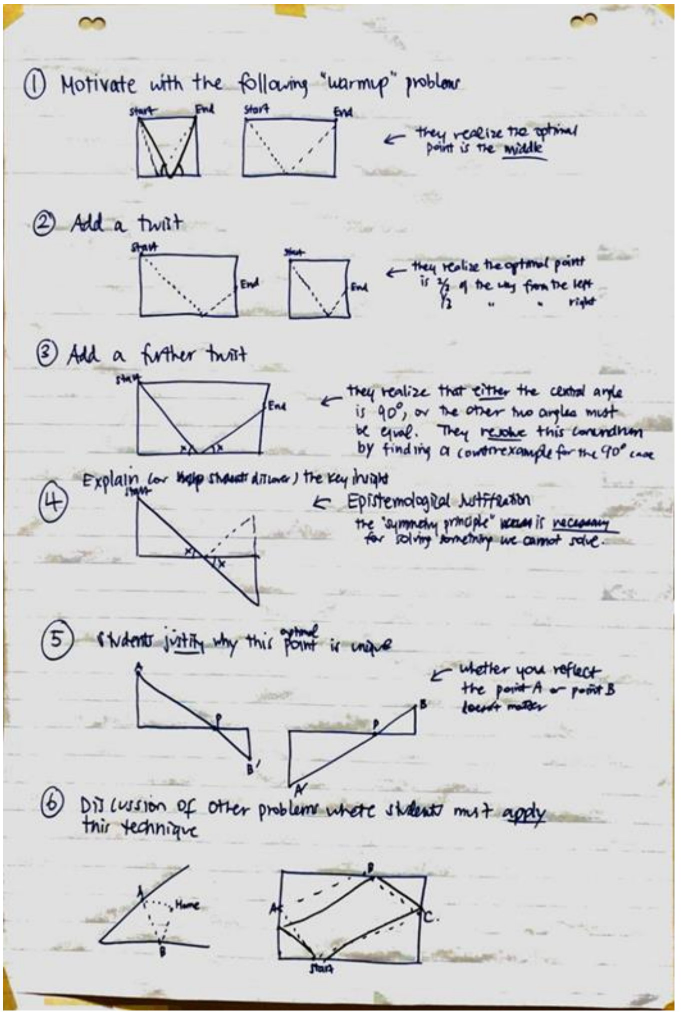

Figure 6

Here were some of our conclusions from Day 2:

- It is important for students to be stuck and realize the limitation of their knowledge.
- It is okay for teachers to propose new ideas to get students 'unstuck' in the problem-solving process.
- The way in which the problem-solving process is managed in class is sometimes more important than the problem itself.

DAY THREE

Here was our focus for the third day: *Discuss how problem solving should be addressed with future educators: in our methods courses, should we explicitly teach problem-solving heuristics and strategies, or perhaps teach content that incorporates problem solving?*

In thinking about the didactical analysis from the previous day, we realized the importance of making things complex, not complicated. Complexity is good for student learning, and there are two types: mathematical complexity (referring to the level of content) and procedural complexity (referring to the number of steps required to solve the problem).

Peter/Ann/Caroline/Nadine/Jimmy shared a poster with us that illustrated this point (See Figure 7). While this poster was created at the end of Day 2, it was fitting that Day 3 began with this discussion, because it tied so well to our focus.

We need to have a balance between task complexity and student ability: once these two items are in balance, we create *flow* (Csíkszentmihályi, 1990).

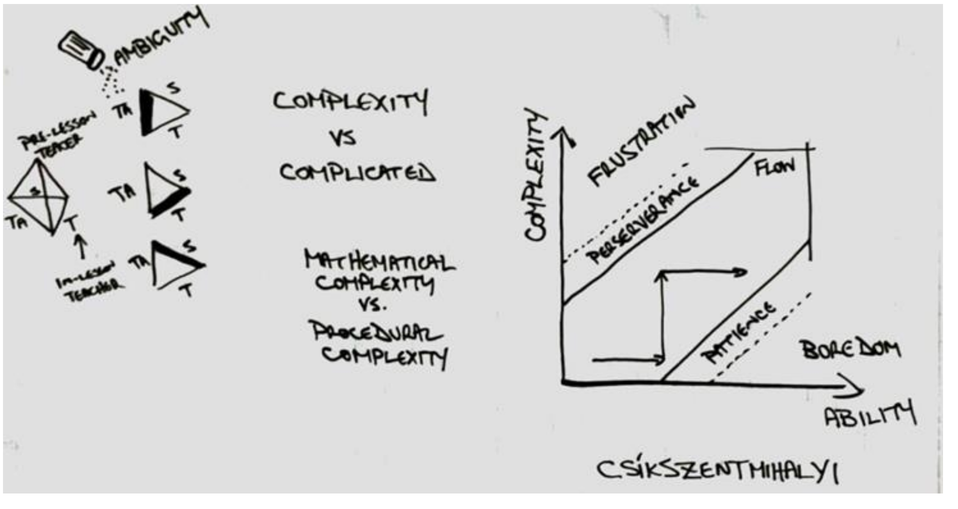

Figure 7

If the ability is too high and the complexity is too low, then the student is bored. If the ability is too low and the complexity is too high, then the student is frustrated. We want to make problems at just the right level of complexity to ensure flow. We want to introduce doubt when a student has solved the problem quickly: Are you sure this is the right answer? This way, students figure out the difference between thinking an answer is true, and knowing that an answer must be true. In other words, they have not just found an *answer*: they have obtained a *solution*.

As for who makes the complexity, sometimes it is the task itself, especially if the task (or problem) is easily understandable and broad. Sometimes it is the teacher who provides the complexity, asking provocative questions that lead to further thinking (e.g. What if you change the taxman problem from $N = 12$ to $N = 18$?) And sometimes it is the student who provides the complexity, asking themselves the questions that inspire further thinking and learning.

There are three approaches of mathematics education: teaching math techniques (e.g. basic skills) so that students can solve problems later; teaching problem-solving heuristics (e.g.

Polya's four techniques); teaching math through problem-solving. The ideal is the third one: we can and should use problem-solving to help students develop mathematical knowledge and to master it.

Many teachers have the (incorrect) belief that students cannot do problem solving unless they have been drilled on technique and content. As a result, students often do 'content' from September to May and then do 'problem solving' in June. Instead, let's integrate the two.

At this point, David asked us the following questions:

- What do we want teachers to know about what a 'problem' is?
- What do we want teachers to do with these problems?
- What do we want teachers to know about problem solving?
- How can they learn this? Through which problems?

The ensuing discussion gave rise to many observations. The following list gives a flavour of the discussion without quite capturing its diversity:

- We seek to create a need for a student to want to learn something, and a carefully-posed problem can be a way to do that. For example, asking students to figure out 28×40 in a Grade 4 class gets students to do it the hard way (for example adding 40 twenty-eight times) and then motivates the value of learning a method (or algorithm) for two-digit by two-digit multiplication.
- We need to recognize that many teachers do not want to do open-ended problem solving because it necessitates losing control, and might make the teacher look bad in front of their students—say if they are surprised with an approach they cannot 'manage' because they have not seen it before. But let's remember that vulnerability is necessary, and there is so much value to making mistakes, modelling lifelong learning. Struggle builds perseverance and character, and being comfortable with being uncomfortable.
- We want teachers to see that problem solving builds confidence, creativity, critical thinking skills, oral communication skills, and written communication skills. These five skills are so important in every child's life, no matter what endeavours he/she chooses to pursue in the future. And mathematical problem solving is arguably the best way to develop these five skills in a student.
- Let's fight the myth that teaching problem solving takes longer. Yes, some problems take longer, especially at the beginning. But if we build "thinking classrooms" (Liljedahl, 2015), then students learn content in a fraction of the time and retain the material better.
- We cannot just teach problem-solving techniques (e.g. heuristics). We need to move into much more complex problems that inspire, contextualize and motivate students to learn mathematical content. But let's remember that high school teachers teach for 200 days, six classes a day. We cannot make every class super-innovative, and it is unrealistic to do a deep inquiry-level class every day.
- We need to build problem-solving communities, which could mean at the local level (e.g. several teachers from the same school working together) or a national community like this where we have intentional discussions about what we are teaching and why, and share resources.
- Immersion, experience, and experimentation are key. We want to immerse teachers in problem-solving environments where they gain experience at solving problems themselves. But we cannot stop there. Teachers need to experiment with trying different problems, seeing which ones work in their classrooms and which ones do not, to continually refine their practice.

- We discussed the Sherlock Holmes Bicycle Tire problem, where a student can use tangent lines to figure out which direction a bicycle was going. Instead of using this as the Keystone Problem, i.e., the problem that is presented to the students after they have learned tangent lines, we can do this at the beginning—where they get to a point where they get stuck, and then they need to learn something new in order to progress further. Once they do, deep learning occurs because of this epistemological justification, much more so than if one had 400 exercises.
- In chaos theory, the most important things happen at the edge of chaos, never in the chaos, and never in a perfectly-ordered system. That applies in our teaching. Let's think about "how do we best teach problem solving?": as a problem to be solved! And let's try to solve this problem in creative and innovative ways.

We concluded the working group by discussing how we could support teachers and students in schools. We realized that one way we can do this is by having moments of 'productive disruption' together, where professors partner with local teachers to co-teach classes together, which creates community.

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