Supporting Mathematical Creativity Through Problem Solving

Richard Hoshino

Abstract I teach at a small Canadian liberal arts and sciences university, where I offer a course called *Mathematical problem-solving*. In this course, undergraduate students develop the four key takeaways of a liberal arts education: critical thinking, creativity, oral communication, and written communication. As a teacher of mathematics, I am biased in my belief that mathematics develops these four takeaways (or skills) in a way that no other subject can. For many years, the core of my teaching practice has been developing these four skills in my students, through carefully-chosen problems ranging from logic puzzles to contest questions.

I offer several problems in this chapter, to illustrate how "applied problem solving" can develop creativity in our students. Specifically, these problems develop a key problem solving strategy or skill, the ability to solve hard problems by converting them into equivalent simpler problems.

I believe that this skill is not just essential for post-secondary students; if we can foster this mathematical problem solving ability in our secondary students, perhaps we could inspire more students with the message that mathematics is beautiful and powerful and relevant to everything in this world: challenging strong students who find classroom mathematics too easy and irrelevant while motivating weaker students who would see that mathematics is accessible, and has important applications to the issues they care about.

The problems in this chapter are based on simple ideas, but reveal surprising connections to deep mathematical ideas that are taught at the undergraduate level, including graph colourings and combinatorial enumeration.

Keywords Creativity · Problem solving · Symmetry · Enumeration · Graph colouring

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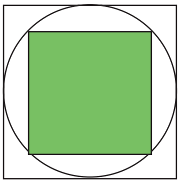
I teach at a small liberal arts and sciences university in Squamish, British Columbia, where I offer a course called *Mathematical problem-solving*. In this course, undergraduate students develop the four key takeaways of a liberal arts education: critical thinking, creativity, oral communication, and written communication. As a teacher of mathematics, I am biased in my belief that mathematics develops these four takeaways (or skills) in a way that no other subject can. For many years, the core of my teaching practice has been developing these four skills in my students, through carefully-chosen problems ranging from logic puzzles to contest questions.

I acquired this pedagogical viewpoint through a lifetime of studying mathematics, which has enabled me to make tangible impacts to my community: creating a new risk-scoring algorithm for high-risk marine cargo; reducing wait times at the Canadian border; serving as the mathematics consultant for three Canadian TV game shows; helping a billion-dollar professional baseball league design a schedule to cut down on greenhouse gas emissions; working with local organizations and companies to manage their staff scheduling; and implementing a roommate-matching program and course registration system at my university. In the process of solving these real-life problems, I have realized that a deeper skill was involved, a skill that I have worked hard to cultivate in my teaching practice: *the ability to solve hard problems by converting them into equivalent simpler problems*.

I believe that this skill is not just essential for post-secondary students; if we can foster this mathematical problem solving ability in our secondary students, perhaps we could inspire more students with the message that mathematics is beautiful and powerful and relevant to everything in this world: challenging strong students who find classroom mathematics too easy and irrelevant while motivating weaker students who would see that mathematics is accessible, and has important applications to the issues they care about.

I offer several problems in this chapter, to illustrate how "applied problem solving" can develop this key skill in our students, to recognize when a hard problem can be converted into a problem that is both equivalent and simpler.

There are numerous solutions to the problem as shown in the below figure. Readers are encouraged to try the problem before reading further!



Example Problem One: *In the diagram, a circle is inscribed in a (large) square, and a (small) square is inscribed in the circle. What is the ratio of the areas of the two squares?*

One approach is to let the small square have side length 1, and show that the side length of the large square must be $\sqrt{2}$. This can be done by noticing that the side length of the large square must equal the diameter of the circle, which must equal the diagonal of the small square, which must be $\sqrt{2}$ by the Pythagorean Theorem.

This proves that the ratio of the two areas must be $\sqrt{2} \times \sqrt{2} = 2$.

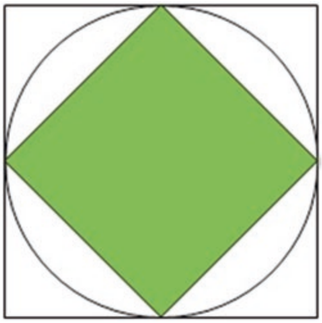


Fig. 1 By rotating the inner square 45 degrees, we develop a key insight

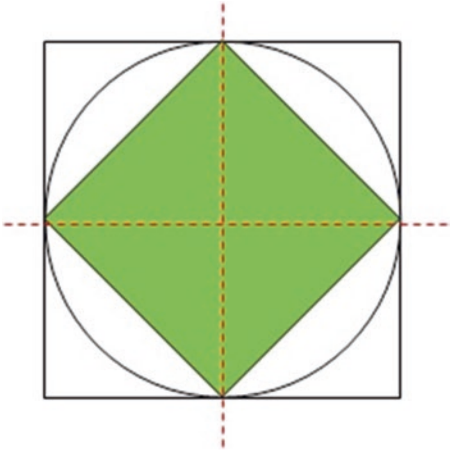


Fig. 2 A visual proof that the outer square is double the area of the inner square

But there is a cleaner solution, once we realize that we can solve this problem by converting it into an equivalent simpler problem. The key is to *recognize and exploit rotational symmetry*.

We rotate the inner square 45 degrees clockwise (Fig. 1).

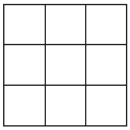
We then draw a vertical line and a horizontal line passing through the centre of the circle (Fig. 2).

Once we have done this, we can quickly see that the area of the large square has to be twice the area of the small square!

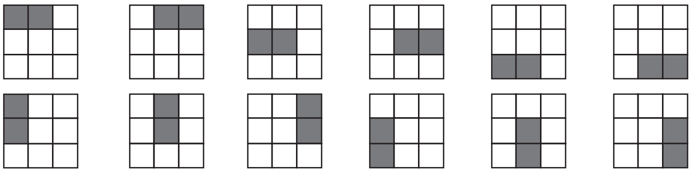
This beautiful solution illustrates that the heart of mathematics is not about memorizing formulas or rules, but rather about problem solving and detecting patterns, insight, and structure to uncover truth. Most of us assume that the inner square has to remain in a fixed position, while we can develop a better solution by breaking a self-imposed constraint, to use a simple idea (of a 45 degree rotation) to convert a difficult problem into one that is surprisingly simple and elegant as shown in the below figure

Example Problem Two:

There are many squares and rectangles, of all sizes, that appear in a 3x3 grid.



For example, there are twelve 1x2 rectangles: six horizontal and six vertical.



How many squares and rectangles, of all sizes, appear in this diagram?

Again, readers are encouraged to solve the problem themselves before reading further. As a challenge, once you have solved the problem in one way, can you think of another method?

One natural approach to solving this problem is to enumerate all possible cases, where we consider squares and rectangles of all possible dimensions, to get the correct answer of 36 (as shown in the below table).

Case	# of squares/rectangles
1 × 1 squares	9
1 × 2 rectangles	12
1 × 3 rectangles	6
2 × 2 squares	4
2 × 3 rectangles	4
3 × 3 squares	1
Total	36

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But if we do this, we miss out on the underlying structure. Here is an equivalent solution, where we consider the dimensions (length and height) of each square and rectangle (as shown in the below table).

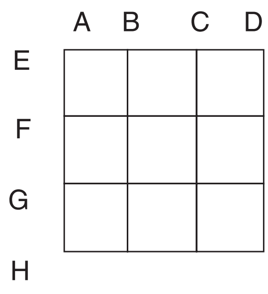
Length	Height	# of squares/ rectangles
1	1	9
1	2	6
1	3	3
2	1	6
2	2	4
2	3	2
3	1	3
3	2	2
3	3	1
Total		36

From here, we notice that the answer is 9 + 6 + 3 + 6 + 4 + 2 + 3 + 2 + 1, which is equivalent to (9 + 6 + 3) + (6 + 4 + 2) + (3 + 2 + 1) = 3(3 + 2 + 1) + 2(3 + 2 + 1) + 1(3 + 2 + 1) = (3 + 2 + 1)(3 + 2 + 1).

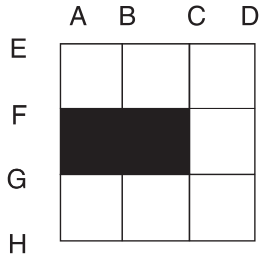
Surely there is a reason for such a beautiful answer. Take a moment to ponder this!

Indeed there is a reason. Instead of enumerating all possible cases, notice that every square or rectangle in our 3 × 3 grid consists of two vertical sides and two horizontal sides. Once we choose our two vertical sides and two horizontal sides, our square or rectangle is uniquely determined.

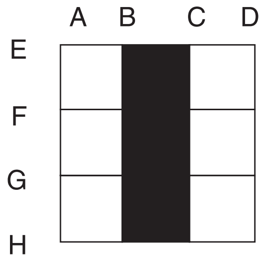
Specifically, what we do is look at each line in our 3 × 3 grid. We label our four vertical lines with the letters A, B, C, D, and label our four horizontal lines with the letters E, F, G, H shown in the below figure.



Notice that each selection of two vertical lines and two horizontal lines traces out a unique square or rectangle. For example, if we select the two vertical lines A and C, as well as the two horizontal lines F and G, then the four lines intersect to form this 1 × 2 rectangle (as shown in the below figure).



Conversely, given any square or rectangle, we can uniquely map it to the selection of two vertical lines and two horizontal lines by extending the sides until it hits the labels. For example, this 3 × 1 rectangle must map to the vertical lines B and C, and the horizontal lines E and H (as shown in the below figure).



By making this mapping, we have in fact constructed an equivalent problem which might be useful. In other words, we have shown that the problem of counting the number of squares and rectangles in a 3 × 3 grid is completely identical to the simpler problem in as shown in the below figure.

Example Problem Three:

Mrs. Smith has received four free tickets to a Justin Bieber concert, and decides to give them away: to two of her female friends, and two of her male friends.

Her female friends are Alice, Bethany, Chamique, and Diana.

Her male friends are Edwin, Fernando, George, and Harry.

Determine the number of different ways Mrs. Smith can give out the four tickets.

Do you see how this is both simpler and equivalent to the previous problem of counting squares and rectangles? Since there are six ways of choosing her female friends (AB, AC, AD, BC, BD, CD) and similarly six ways of choosing her male

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friends, the correct answer must be $6 \times 6 = 36$. In fact, this use of the Fundamental Counting Principle may be familiar to intermediate students, if they have completed a unit in elementary probability.

Thus, to solve the problem of determining the number of squares and rectangles in a 8×8 checkerboard grid, we do not need to enumerate all possible cases; instead, what we do is convert this problem into an equivalent simpler problem, the above "Ticket Problem" where Mrs. Smith has nine female friends and nine male friends. Can you see why the 8×8 checkerboard relates to the scenario with *nine* friends of each gender?

In this problem, the correct answer is $36 \times 36 = 1296$, since we can show that there are 36 ways that Mrs. Smith can choose two out of the nine females, and 36 ways that she can choose two out of the nine males. This is a much more elegant solution than manually enumerating all the cases.

Please explore the problem as shown in the below figure on your own or in a small group before reading on.

Example Problem Four:

Several students have formed various clubs, based on academic subjects that most interest them. The clubs consist of the following students:

Astronomy Club: Michael, Breanna, Joe

Biology Club: Breanna, Bonnie

Calculus Club: Joe, Caitlin

Dance Club: Joe, Gillian, Patrick

Economics Club: Caitlin, Michael, Bonnie

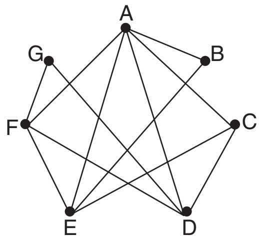
Food Studies Club: Bethany, Gillian, Michael

Geology Club: Bethany, Patrick

Each of the seven clubs wants to have an hour-long meeting on Friday afternoon; each person in the club must be present for the meeting. Class ends at noon, and the eight students want to get their club meetings over with as soon as possible. What is the earliest possible time at which all eight students can complete each of their hour-long meetings?

In your work, you might have noticed that the most inefficient solution is 7:00 PM, by having each of the seven clubs meeting in hour-long slots, one right after the other. We can save time by combining slots where no conflict occurs: for example, having the Calculus and Geology clubs meeting at the same time, since no student belongs to both clubs. This produces a solution whose answer is 6:00 PM.

Fig. 3 Representing the problem as a conflict graph



Eventually, students find the correct answer of 3:00 PM by determining a way that the seven clubs can meet in 3 h-long slots, through trial and error. For example, here is one possible 3-h solution:

12 PM to 1 PM: Astronomy, Geology
1 PM to 2 PM: Economics, Dance
2 PM to 3 PM: Food Studies, Biology, Calculus

A common approach is to make an 8 × 7 table with students' names in the rows and clubs in the columns, to see where the possible conflicts arise. Through such a table, students determine which clubs can meet together and which ones cannot, and find a solution such as the one above.

Once students find a solution for 3 h, they are asked whether there exists a solution for 2 h. They quickly see that no 2-h solution exists, since Michael and Joe each belong to three different clubs: since each of them needs at least 3 h to complete their meetings, the entire group needs at least 3 h as well.

Despite satisfaction with solving the problem, students remark that making a 56-element table is tedious and lengthy. They realize that all is required is to determine which clubs have conflicts—e.g., Astronomy and Biology cannot meet at the same time because one individual belongs to both clubs: what matters is that there is an individual belonging to both clubs, not who that individual is.

Through this process, we motivate the key idea of solving scheduling problems using graph theory, to show that the above scheduling question can be solved by creating a "conflict graph" on seven vertices (representing the seven clubs denoted by the letters A, B, C, D, E, F, G), where two vertices are joined by an edge if and only if some individual belongs to both clubs, and would therefore have a conflict if both clubs scheduled their meetings at the same time.

For this particular problem, the conflict graph has seven vertices and twelve edges, and looks like Fig. 3.

Now we *colour* the vertices, with each colour representing a time slot. For example, if we colour vertex A red, then we see that we cannot assign red to vertices B, C, D, E, F, since each of those five clubs has a conflict with vertex A. Thus, we must use a different colour.

What is the fewest number of colours we need to ensure that no edge is joined by two vertices of the same colour? Do you see how the answer to this question must be the same as the fewest number of time slots needed to schedule all the student clubs?

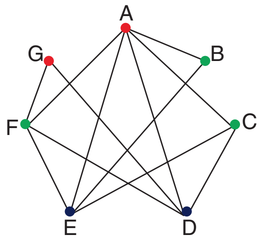


Fig. 4 Our solution, with red={A,G}, blue={D,E}, green={B,C,F}

We show that only three colours are required. To do this, we let A be red, E be blue, and F be green. Then, B and C must be green (since they are both adjacent to the red vertex A and the blue vertex E), which in turn forces D to be blue and G to be red. Therefore, we have found a valid 3-colouring (Fig. 4).

In the above picture, we see that no edge connects two vertices of the same colour. Thus, we are guaranteed a solution to the seven-club scheduling problem by simply assigning time slots to the three colours: Red = 12 PM–1 PM, Blue = 1 PM–2 PM, and Green = 2 PM–3 PM. Indeed, we can quickly verify that this is the exact same solution as what was given earlier.

A natural question is whether two colours suffice. To see why this is impossible, note that AEF is a triangle, representing the three different clubs Michael is in. And so each of these three points must be assigned different colours; thus we require at least three colours, i.e., at least three time slots.

Many mathematical problems can only be solved in routine and mundane ways. However, if secondary students see problems that can be solved in *both* a routine way and a surprising innovative way, then many unexpected benefits arise: a greater confidence in doing mathematics, a deeper appreciation for the beauty of mathematics, a development in one's creativity, as well as the opportunity to engage in applied problem solving. These skills and opportunities would help secondary students in so many ways, and serve them well for their future.

Various resources, geared towards secondary students and their teachers, are available at www.richardhoshino.com. Please use whatever you wish, free of charge.

Additional Suggestions for Further Reading

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