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THE EDGE DENSITY OF CRITICAL DIGRAPHS

RICHARD HOSHINO, KEN-ICHI KAWARABAYASHI

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Let $\chi(G)$ denote the chromatic number of a graph G. We say that G is k-critical if $\chi(G)=k$ and $\chi(H)<k$ for every proper subgraph $H\subset G$. Over the years, many properties of k-critical graphs have been discovered, including improved upper and lower bounds for $||G||$, the number of edges in a k-critical graph, as a function of $|G|$, the number of vertices. In this note, we analyze this edge density problem for directed graphs, where the chromatic number $\chi(D)$ of a digraph D is defined to be the fewest number of colours needed to colour the vertices of D so that each colour class induces an acyclic subgraph. For each $k\geq 3$, we construct an infinite family of sparse k-critical digraphs for which $||D||<(\frac{k^2-k+1}{2})|D|$ and an infinite family of dense k-critical digraphs for which $||D||>\left(\frac{1}{2}-\frac{1}{2^{k-1}}\right)|D|^2$. One corollary of our results is an explicit construction of an infinite family of k-critical digraphs of digirth l, for any pair of integers $k,l\geq 3$. This extends a result by Bokal et al. who used a probabilistic approach to demonstrate the existence of one such digraph.

1. Introduction

Let G be an undirected graph with no loops or multiple edges. Let $\chi(G)$ denote the *chromatic number* of G, i.e., the fewest number of colours needed to colour the vertices of G so that each colour class induces an independent set.

A graph G is k-critical if $\chi(G)=k$ and $\chi(H)<k$ for every proper subgraph $H\subset G$. For example, the complete graph K_k is k-critical for all $k\geq 1$, and every odd cycle C_{2n+1} is 3-critical.

Originally introduced by Dirac [5] nearly sixty years ago, critical graphs are important as they represent the family of minimal k-chromatic graphs;

Mathematics Subject Classification (2000): 05C20, 05C35, 05C40 indeed, finding a small k-critical subgraph of G gives a certificate for $\chi(G) \geq k$. Often problems for k-chromatic graphs can be reduced to problems for k-critical graphs, which have a simpler and more restrictive structure.

Let $|G|$ and $||G||$ respectively denote the number of vertices and number of edges in G. If G is k-critical, then we have the trivial inequality $||G|| \geq \frac{|G|(k-1)}{2}$, since a k-critical graph must have minimum degree $k-1$.

The *edge density problem* of determining tight upper and lower bounds on $||G||$ as a function of $|G|$ has been the subject of much research among graph theorists. Even for $k=4$, the optimal bounds are unknown, and there has been little progress on developing a characterization theorem for the family of 4-critical graphs. For $k>4$, the problem becomes significantly harder.

In this paper, we study the k-criticality of *directed* graphs D, motivated by recent research in this "new" area of graph theory (c.f. [14]). We determine upper and lower bounds for the edge density of k-critical digraphs, using the definition $\chi(D)$, where each colour class induces an acyclic subgraph rather than an independent set. In our analysis, we introduce two constructions that generate infinite families of k-critical digraphs, including those with arbitrarily large digirth. This extends a result by Bokal et al. [3] who used a probabilistic approach to prove the existence of *one* digraph D with $\chi(D) \geq k$ and digirth $\geq l$, for all $k, l \geq 3$.

We proceed as follows. In Section 2, we provide the known results for the edge density of k-chromatic *undirected* graphs, and in Section 3, we formally define the chromatic number of *directed* graphs and provide several examples of 3-critical digraphs. In Section 4, we present a recursive construction that generates a k-critical digraph from several $(k-1)$-critical digraphs connected via a root vertex, and use this construction to produce an infinite family of *dense* k-critical digraphs satisfying $||D|| > \left(\frac{1}{2} - \frac{1}{2^{k-1}}\right)|D|^2$, for each $k \geq 3$. In Section 5, we analyze properties of circulant tournaments and apply a variant of the Hajós construction to generate an infinite family of *sparse* k-critical digraphs satisfying $||D|| < \left(\frac{k^2-k+1}{2}\right)|D|$ for each $k \geq 3$. In Section 6, we conclude with two open problems that ask whether these bounds are optimal.

2. Edge density of undirected graphs

For any n, let $f_k(n)$ and $g_k(n)$ respectively denote the minimum and maximum number of edges that can appear in a k-critical graph on n vertices.

Trivially, we have $f_1(n) = g_1(n) = 0$ and $f_2(n) = g_2(n) = 1$, since K_1 is the only 1-critical graph and K_2 is the only 2-critical graph. Since the only 3-critical graphs are the odd cycles, we have $f_3(n) = g_3(n) = n$ for all odd n.

For $k \geq 4$, Gallai [8] conjectured that

$$f_k(n) \geq \frac{n(k-1)}{2} + \frac{(k-3)(n-k)}{2(k-1)}.$$

For example, if $k=4$, the above inequality corresponds to $f_4(n) \geq \frac{5n-2}{3}$. Even this seemingly simple lower bound is still unresolved; the best known result is a recent proof by Farzad and Molloy [7] that this inequality is true provided that the subgraph induced by vertices of degree 3 is connected.

In 2003, Gallai's lower bound was improved [13], showing that for $k \geq 6$, every k-critical graph other than K_k has $f_k(n) \geq \frac{n(k-1)}{2} + \frac{(k-3)n}{2((k-c)(k-1)+k-3)}$, where $c=(k-5)(\frac{1}{2} - \frac{1}{(k-1)(k-2)})$.

A recent result [14] has found a better lower bound $f_k(n) \geq F(k,n)$ that is sharp for every $n \equiv 1 \pmod{k-1}$, as well as for $k=4$ and every $n \geq 6$. This result by Kostochka and Yancey establishes the asymptotics of $f_k(n)$ for every fixed k and proves a conjecture by Ore from 1967 that for every $k \geq 4$ and $n \geq k+2$, the identity $f_k(n+k-1) = f_k(n) + \binom{k-1}{2}\left(k - \frac{2}{k-1}\right)$ holds for each $k \geq 4$ for all but at most $\frac{k^3}{12}$ values of n.

For the upper bound, Toft [19] proved that for every $k \geq 4$, there exists a positive constant c_k such that $g_k(n) > c_k n^2$ for all n with $n \geq k$ and $n \neq k+1$. Furthermore, Toft obtained the specific bounds $c_4 \geq \frac{1}{16}$ and $c_5 \geq \frac{4}{31}$.

By completely joining two disjoint odd cycles of equal length to produce a 6-chromatic graph [5], we see that $g_6(n) \geq \frac{n^2}{4} + n$. It remains an open problem [12] whether the constants $\frac{1}{16}$, $\frac{4}{31}$, and $\frac{1}{4}$ are optimal for the cases $k=4$, 5, and 6, respectively.

Note that by Turán's Theorem [20], any graph with more than $\frac{k-2}{2(k-1)}n^2$ edges contains K_k as a subgraph. Thus, $c_k \leq \frac{k-2}{2k-2}$ is a trivial upper bound for the function c_k. Surprisingly, this remains the best known upper bound [11].

3. The chromatic number of digraphs

In a digraph (directed graph) D, edge uv is the arc from the initial vertex u to the terminal vertex v. We say that a vertex set $A \subseteq V(D)$ is *acyclic* if the induced subgraph (subdigraph) $D[A]$ has no directed cycles. The chromatic number $\chi(D)$ of a digraph D is defined the same way as $\chi(G)$, except that each colour class must induce an acyclic subgraph rather than an independent set. A digraph D satisfies $\chi(D) = 1$ iff D contains no directed cycle.

The chromatic number of digraphs was originally defined by Neumann-Lara [15] as the *dichromatic* number, and was introduced independently two decades later by Mohar [17] through his analysis of the circular chromatic number of weighted graphs. Just as the girth of a graph G is the length of the shortest cycle in G, the *digirth* is the length of the shortest directed cycle in D.

The chromatic number of digraphs makes important connections and applications to various branches of discrete mathematics. For example, consider the problem of characterizing the set of graphs H for which all graphs not containing H as an induced subgraph have bounded chromatic number. As explained by Berger et al. [2], this problem is trivial for graphs (the only such graphs are K_1 and K_2) but extremely complex for *tournaments*, i.e., complete digraphs. By considering $\chi(D)$ instead of $\chi(G)$, we can develop a rich theory of tournament colouring, and generalize the Ramsey-type theorems of Erdős and Hajnal [6].

While it may not be intuitive why we define $\chi(D)$ as a partition into acyclic subgraphs, we find that this is the most natural way of extending the theories of undirected graph colourings so that they also apply in the context of directed graphs. For example, under this definition, results on Gallai colourings and list colourings also generalize to digraphs [9]. And as proved by Harutyunyan and Mohar [10], there exist digraphs of maximum degree Δ and of arbitrarily large digirth whose chromatic number is at least $\frac{c\Delta}{\log \Delta}$, thus generalizing a result of Bollobás [4] for undirected graphs. Finally, there is a fascinating connection [18] between $\chi(D)$ and its spectral radius (i.e., the largest modulus of an eigenvalue of the adjacency matrix $A(D)$), thus showing that the relationship between the chromatic number of a graph and its eigenvalues [21] also applies to digraphs.

Note that if we replace each edge of graph G with a pair of oppositely-directed edges joining the same pair of vertices, then $\chi(D) = \chi(G)$ since any two adjacent vertices in D induce a *digon*, a trivial directed cycle of length 2. Note that if D is a digraph produced from G by randomly orienting the edges, then any k-colouring of G is automatically a k-colouring of D, i.e., $\chi(D) \leq \chi(G)$.

Surprisingly, it is often harder to compute $\chi(D)$ than $\chi(G)$. To cite one example, it is trivial to check whether a graph G has chromatic number at most 2. However, the equivalent decision problem for digraphs is *NP*-complete [3].

As in the case of undirected graphs, we say that a digraph D is k-critical if $\chi(D) = k$ and $\chi(H) < k$ for every proper subgraph (subdigraph) $H \subset D$. In other words, for any k-critical digraph D, the removal of any edge e allows the digraph to be properly $(k-1)$-coloured. For example, it is a routine exercise to confirm that D is 2-critical iff D is a directed cycle.

It is straightforward to show [18] that if D is k-critical, then each vertex $v \in V(D)$ must have total degree at least $2k-2$, as both the in-degree and out-degree of v must be at least $k-1$. Letting $\overrightarrow{f_k}(n)$ denote the minimum number of edges that can appear in a k-critical digraph on n vertices, we have $\overrightarrow{f_k}(n) \geq (k-1)n$. Note that we can attain this trivial lower bound by taking the *bi-directed* complete graph K_k that joins each pair of vertices with two oppositely-directed edges. And for $k=3$, we can construct infinitely many digraphs with $\overrightarrow{f_3}(n) = 2n$ by simply taking the family of bi-directed odd cycles C_n.

As a result, the edge density problem is uninteresting when allowing digons. Thus, we will only allow digon-free directed graphs, where each pair of vertices is connected by at most one edge, i.e., $uv \in E(D) \to vu \notin E(D)$.

A lengthy case-analysis [16] shows that there are only four (digon-free) 3-chromatic digraphs with $|D|=7$ and $||D||=21$. Two of these 3-chromatic digraphs are 3-critical, while the other two are not as there exists an edge whose removal preserves the chromatic number. Removing any one of these "special" edges produces a 20-edge digraph, and a computer check shows that all such digraphs are 3-critical *and* are pairwise isomorphic. Figure 1 lists the three 3-critical digraphs with $|D|=7$, where digraphs (a) and (b) have $||D||=21$ edges and digraph (c) has $||D||=20$ edges.

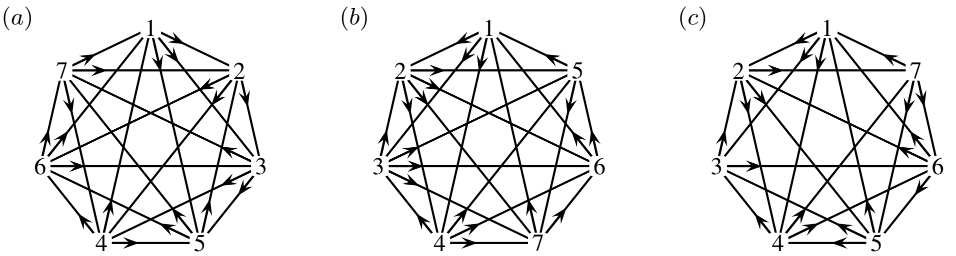

Figure 1. The set of 3-critical digraphs on 7 vertices.

While we have a full characterization of all 3-critical digraphs on 7 vertices, we do not have a characterization of 3-critical digraphs for any $|D|>7$. Even for $|D|=8$, a computer search generates dozens of non-isomorphic 3-critical digraphs, with $21 \leq ||D|| \leq 25$.

Alas, it is not clear whether there is a general structure of 3-critical digraphs, even for the case $|D|=8$. This motivates the study of edge density, to see if we can infer any patterns on the structure of k-critical digraphs by finding optimal upper and lower bounds for $||D||$ as a function of $|D|$.

Ideally, we would like to prove whether there exist functions $x(k)$ and $y(k)$ such that for every $k \geq 3$ there are infinitely many k-critical digraphs satisfying $||D|| > x(k)|D|^2$, and infinitely many k-critical digraphs satisfying $||D|| < y(k)|D|$. In the next two sections, we will prove the following theorems, which demonstrate the existence of these functions $x(k)$ and $y(k)$.

Theorem 1. *For each $k \geq 3$, there exists an infinite family of dense k-critical (digon-free) digraphs for which $||D|| > \left(\frac{1}{2} - \frac{1}{2^{k-1}}\right)|D|^2$.*

Theorem 2. *For each $k \geq 3$, there exists an infinite family of sparse k-critical (digon-free) digraphs for which $||D|| < \left(\frac{k^2-k+1}{2}\right)|D|$.*

4. Upper bound – dense digraphs

We first establish Theorem 1. Following the notation of Berger et al. [2], we define $X \to Y$ if every vertex in X is adjacent *to* every vertex in Y. We write $v \to X$ for $\{v\} \to X$, and $Y \to v$ for $Y \to \{v\}$.

For a fixed integer $l \geq 3$, define $D^* := (v, D_1, D_2, \ldots, D_{l-1})$ to be the digraph constructed from a single root vertex v and $l-1$ different digraphs $D_1, D_2, \ldots, D_{l-1}$, where $v \to D_1$, $D_i \to D_{i+1}$ for $1 \leq i \leq l-2$, and $D_{l-1} \to v$. We say that such a digraph is a *rooted l-section*.

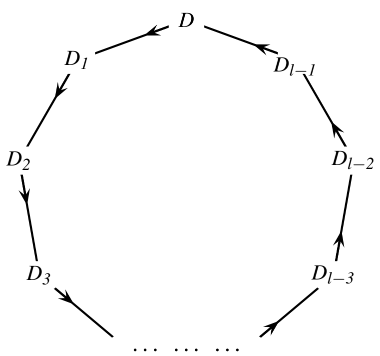

Figure 2. Construction of a rooted l-section, where D_i is a digraph for each $1 \leq i \leq l-1$.

For example, the rooted 3-section $(v, \overrightarrow{C_3}, \overrightarrow{C_3})$ appears as digraph (b) in Figure 1, with $v = \{1\}$, $D_1 = \{2,3,4\}$, and $D_2 = \{5,6,7\}$.

We now prove the following result, which generates infinitely many k-critical graphs for any $k \geq 3$. Recall that for any digraph D, the *digirth* $l := l(D)$ is the length of the shortest directed cycle in D.

Proposition 1. *For some $n \geq 3$, let $D_2 = \overrightarrow{C_n}$, the directed cycle on n vertices. For each $k \geq 3$, define*

$$D_k := (v, D_{k-1}, D_{k-1}, \ldots, D_{k-1})$$

to be the rooted l-section where each of the $l-1$ digraphs is a copy of D_{k-1}. Then D_k is k-critical if D_{k-1} is $(k-1)$-critical. Moreover, D_k has digirth l if D_{k-1} has digirth $\geq l$.

Proof. D_k consists of a root vertex v and $l-1$ copies of the digraph D_{k-1}. Label these l–1 copies $X_1, X_2, \ldots, X_{l-1}$. Furthermore, for each $1 \leq i \leq l$–1, let $V(X_i) = \{x_{i,1}, x_{i,2}, \ldots, x_{i,m}\}$ be the vertices of digraph X_i, where $m = |D_{k-1}|$. We now prove that if D_{k-1} is $(k-1)$-critical, then D_k is k-critical.

Since X_i is $(k-1)$-critical, there exists a function $\phi_i \colon V(X_i) \to \{1, 2, \ldots, k-1\}$ that is a $(k-1)$-colouring of X_i. Define the function $\Phi(v) = k$ and $\Phi(x_{i,j}) = \phi_i(x_{i,j})$ for all pairs (i, j) with $1 \leq i \leq l-1$ and $1 \leq j \leq m$. Then this function $\Phi \colon V(D_k) \to \{1, 2, \ldots, k\}$ is a k-colouring of D_k since $\Phi^{-1}(c)$ induces an acyclic subgraph for each colour $1 \leq c \leq k$. Thus, $\chi(D_k) \leq k$.

Now suppose $\chi(D_k) < k$. Then there exists a function $\Phi^* \colon V(D_k) \to \{1, 2, \ldots, k-1\}$ that is a $(k-1)$-colouring of D_k. Then $\Phi^*(v) = c$ for some $1 \leq c \leq k$–1. Since each X_i is $(k$–$1)$-critical, there must be at least one vertex $x_{i,\overline{j_i}}$ with $\Phi^*(x_{i,\overline{j_i}}) = c$, for all $1 \leq i \leq l$–1. Then the set $\{v, x_{1,\overline{j_1}}, x_{2,\overline{j_2}}, \ldots, x_{l-1,\overline{j_{l-1}}}\}$ induces a directed l-cycle, contradicting our assumption that Φ^* is a valid $(k-1)$-colouring of D_k. Therefore, we have proven that $\chi(D_k) = k$.

To prove that D_k is k-critical, it remains to show that $\chi(D_k - e) \leq k-1$ for all $e \in E(D_k)$. We have four cases to consider: $e = vx_{1,\hat{j}}$, $e = x_{\tilde{i},\hat{j}}x_{\tilde{i},\check{j}}$, $e = x_{\tilde{i},\hat{j}}x_{\tilde{i}+1,\check{j}}$, and $e = x_{l-1,\hat{j}}v$. By symmetry, the fourth case is equivalent to the first, and so it suffices to address the first three cases. In all three cases, we will use the set of functions $\phi_i \colon V(X_i) \to \{1, 2, \ldots, k-1\}$ that we defined earlier.

Case 1. If $e = vx_{1,\hat{j}}$, let $\Phi(v) = \Phi(x_{1,\hat{j}}) = k-1$. Let $\Phi(x_{i,j}) = \phi_i(x_{i,j})$ for all pairs (i, j) with $2 \leq i \leq l-1$ and $1 \leq j \leq m$. Now let $\overline{\phi}$ be a $(k-2)$-colouring of X_1–$x_{1,\hat{j}}$, which must exist by the $(k$–$1)$-criticality of X_1. Let $\Phi(x_{1,j}) = \overline{\phi}(x_{1,j})$ for all $1 \leq j \leq m$ with $j \neq \hat{j}$. Then Φ is a valid $(k-1)$-colouring of $D_k - e$.

Case 2. If $e = x_{\tilde{i},\hat{j}}x_{\tilde{i},\check{j}}$, let $\Phi(v) = k-1$. Let $\Phi(x_{i,j}) = \phi_i(x_{i,j})$ for all pairs (i, j) with $1 \leq i \leq l$–1, $i \neq \tilde{i}$, and $1 \leq j \leq m$. Now let $\overline{\phi}$ be a $(k$–$2)$-colouring of

$X_{\tilde{i}} - e$, which must exist by the $(k-1)$-criticality of $X_{\tilde{i}}$. Let $\Phi(x_{\tilde{i},j}) = \overline{\phi}(x_{\tilde{i},j})$ for all $1 \le j \le m$. Then Φ is a valid $(k-1)$-colouring of $D_k - e$.

Case 3. If $e = x_{\tilde{i},\hat{j}} x_{\tilde{i}+1,\check{j}}$, let $\Phi(v) = \Phi(x_{\tilde{i},\hat{j}}) = \Phi(x_{\tilde{i}+1,\check{j}}) = k-1$. Let $\Phi(x_{i,j}) = \phi_i(x_{i,j})$ for all pairs (i,j) with $1 \le i \le l-1$, $i \neq \{\tilde{i}, \tilde{i}+1\}$, and $1 \le j \le m$. Now let $\overline{\phi_{\tilde{i}}}$ be a $(k-2)$-colouring of $X_{\tilde{i}} - x_{\tilde{i},\hat{j}}$ and $\overline{\phi}_{\tilde{i}+1}$ be a $(k-2)$-colouring of $X_{\tilde{i}+1} - x_{\tilde{i}+1,\check{j}}$, both of which must exist by the $(k-1)$-criticality of $X_{\tilde{i}}$ and $X_{\tilde{i}+1}$. Let $\Phi(x_{\tilde{i},j}) = \overline{\phi_{\tilde{i}}}(x_{\tilde{i},j})$ for all $1 \le j \le m$ with $j \neq \hat{j}$, and $\Phi(x_{\tilde{i}+1,j}) = \overline{\phi}_{\tilde{i}+1}(x_{\tilde{i}+1,j})$ for all $1 \le j \le m$ with $j \neq \check{j}$. Then Φ is a valid $(k-1)$-colouring of $D_k - e$.

This concludes all three cases, and so we have established that D_k is k-critical whenever D_{k-1} is $(k-1)$-critical. To complete the proof, we note that D_k has girth at most l, since the subgraph induced by the set $\{v, x_{1,1}, x_{2,1}, \ldots, x_{l-1,1}\}$ is a directed l-cycle. However, if D_{k-1} has digirth $\ge l$, then clearly there is no r-cycle in D_k for any $r < l$. Hence, D_k must have digirth l as well.

This completes the proof. ∎

For any fixed $l \ge 3$, consider the family of 2-critical digraphs, $D_2 = \overrightarrow{C_n}$ with $n \ge l$. From these 2-critical digraphs, we can apply Proposition 1 to construct an infinite family of k-critical digraphs of digirth l, for any $k \ge 3$. This result is the digraph analogue of a theorem by Erdős, who demonstrated the existence of infinitely many k-critical graphs of arbitrarily high girth [6].

We now use Proposition 1 to generate infinitely many digraphs that are highly dense, which will establish Theorem 1. By the definition of a rooted l-section, $|D_k| = 1 + (l-1)|D_{k-1}|$ and $||D_k|| = (l-1)||D_{k-1}|| + 2|D_{k-1}| + (l-2)|D_{k-1}|^2$.

Since $|D_2| = ||D_2|| = n$, a simple induction argument shows that

$$\begin{aligned} |D_k| &= (l-1)^{k-2}\left(n + \frac{1}{l-2}\right) - \frac{1}{l-2}, \\ ||D_k|| &= \left((l-1)^{2k-5} - (l-1)^{k-3}\right)\left(n + \frac{1}{l-2}\right)^2 \\ &\quad + (l-1)^{k-2}\left(n - \frac{1}{(l-2)^2}\right) + \frac{1}{(l-2)^2}. \end{aligned}$$

We determine the following inequality, which holds true for any $k, n, l \geq 3$:

$$
\begin{aligned}
&||D_k|| - |D_k|^2 \left(\frac{1}{l-1} - \frac{1}{(l-1)^{k-1}} \right) \\
&\quad = n \left[(l-1)^{k-3} \left(l - 1 + \frac{2}{l-2} \right) - \frac{2}{(l-1)(l-2)} \right] \\
&\qquad - \left[\frac{(l-3)(l-1)^{k-3}}{(l-2)^2} - \frac{(l-4)(l-1)^{k-2}+1}{(l-1)^{k-1}(l-2)^2} \right] \\
&\quad > (l-1)^{k-3} \left(l - 1 + \frac{2}{l-2} - \frac{l-3}{(l-2)^2} \right) - \frac{2}{(l-1)(l-2)} \\
&\quad \geq \left(l - 1 + \frac{2}{l-2} - \frac{l-3}{(l-2)^2} \right) - \frac{2}{(l-1)(l-2)} \\
&\quad > 0.
\end{aligned}
$$

Thus, we have shown that for all $k \geq 3$, our digraph D_k satisfies $||D_k|| > \left(\frac{1}{l-1} - \frac{1}{(l-1)^{k-1}} \right) |D_k|^2$ for each k. Since we can build D_k from any initial 2-critical digraph $D_2 = \overrightarrow{C_n}$, for all $n \geq l$, we have generated an infinite family of k-critical digraphs D_k with digirth l satisfying the desired inequality.

Letting $l = 3$, we have proven the existence of an infinite family of dense k-critical digraphs satisfying $||D|| > \left(\frac{1}{2} - \frac{1}{2^{k-1}} \right) |D|^2$. This proves Theorem 1.

5. Lower bound – sparse digraphs

The Hajós construction is a simple procedure that generates infinite families of k-critical graphs, starting with two copies of the trivial k-critical graph K_k. There is an equivalent construction for generating infinite families of k-critical digraphs, which we will use to prove Theorem 2.

Proposition 2. *Given digraphs D_1 and D_2, select an edge u_1v_1 from D_1 and an edge v_2u_2 from D_2. Now create a digraph D from D_1 and D_2 by applying the following three steps: (i) delete u_1v_1 and v_2u_2; (ii) identify the vertices u_1 and u_2; (iii) add the edge v_2v_1. Then D is k-critical, provided both D_1 and D_2 are k-critical.*

We omit the proof as the details are straightforward, and similar to the case analysis presented in Proposition 1. (We trivially note that k-critical graphs do not have vertices of degree 1, and so the new graph D cannot have any isolated vertices.)

We now apply this Hajós digraph construction to establish the following:

Proposition 3. *Let $\tilde{D}$ be a k-critical digraph with v vertices and e edges, with $k \geq 3$. Then there exist infinitely many k-critical digraphs for which $||D|| < \frac{e-1}{v-1}|D|$.*

Proof. Set $D_1 := \tilde{D}$ and $D_2 := \tilde{D}$. By Proposition 2, there exists a k-critical digraph D_3 with $2v-1$ vertices and $2e-1$ edges.

Reapplying Proposition 2 on digraphs D_1 and D_{r-1} for each $r \geq 3$, we generate a digraph D_r with $(r-1)v-(r-2)$ vertices and $(r-1)e-(r-2)$ edges. Then each digraph D in the infinite family $\{D_r\}_{r\geq 1}$ satisfies $||D|| = \frac{e-1}{v-1}|D| - \frac{e-v}{v-1}$.

Since $k \geq 3$, we must have $e > v$, and so $||D|| = \frac{e-1}{v-1}|D| - \frac{e-v}{v-1} < \frac{e-1}{v-1}|D|$. ∎

For example, consider digraph (c) in Figure 1, which is 3-critical with $v=7$ and $e=20$. Then by Proposition 3, there are infinitely many digraphs for which $||D|| < \frac{19}{6}|D|$.

In order to apply Proposition 3, it suffices to find *one* k-critical digraph. Ideally, each of these k-critical digraphs will have a low ratio of edges to vertices so that our lower bound can be made as strong as possible.

To construct a k-critical digraph with a low $\frac{e}{v}$ ratio, we consider complete *tournaments*. While a complete tournament on v vertices has the maximum possible number of edges as a function of v, we show that there exist k-critical tournaments for "small" values of v, implying that for this small value of v, the ratio $\frac{e}{v}$ is at most $\frac{v-1}{2}$. This key insight is what enables us to prove Theorem 2.

To generate this "low-ratio" k-critical digraph, for each k, define the set

$$\begin{aligned} S(k) &:= \{1,2,\ldots,k-1\} \cup \{k+1,k+2,\ldots,2k-2\} \\ &\qquad \cup \{2k+1,\ldots,3k-3\} \cup \ldots \cup \{k^2-2k+1\} \\ &= \bigcup_{j=0}^{k-2} \{jk+1, jk+2, \ldots, (j+1)(k-1)\}. \end{aligned}$$

Now let $C(k)$ be the *circulant* digraph with vertex set $\{1,2,\ldots,k^2-k+1\}$, where edge uv appears in $C(k)$ iff $v-u \in S(k)$, with subtraction taken modulo k^2-k+1.

As an example, $C(3)$ is digraph (a) in Figure 1. Note that for each $1 \leq x \leq k^2-k$, exactly one of x and $(k^2-k+1)-x$ appears in $S(k)$. Thus, the digraph $C(k)$ is a circulant *tournament*, where each pair of vertices is connected by a single edge.

It is a routine exercise [1] to show that k is the maximum cardinality of an acyclic subgraph in $C(k)$, which implies that $\chi(C(k)) \geq \frac{k^2-k+1}{k}$, i.e., that $\chi(C(k)) \geq k$. Furthermore, $C(k)$ can be decomposed into $k-1$ disjoint acyclic subtournaments with k vertices, and a single leftover vertex, thus implying that $C(k)$ is k-chromatic. Specifically, the function $\phi(x) = \lfloor \frac{x+k-1}{k} \rfloor$ for $1 \leq x \leq k^2 - k + 1$ is a valid k-colouring of $C(k)$, which proves that $\chi(C(k)) = k$.

Since any k-chromatic digraph has a k-critical subgraph, there must exist a digraph $\tilde{D} \subseteq C(k)$ with at most k^2-k+1 vertices for which $\tilde{D}$ is k-critical. For $k=3$ and $k=4$, a simple check verifies that $\tilde{D} = C(k)$, i.e., the only k-critical subgraph of $C(k)$ is $C(k)$ itself. Perhaps for larger values of k, there exists a set of edges that can be removed from $C(k)$ to obtain a k-critical graph.

Therefore, for all $k \geq 3$, there exists some k-critical digraph $\tilde{D}$ with $v \leq k^2-k+1$ vertices and e edges, where $e \leq \binom{v}{2} = \frac{v(v-1)}{2}$. By Proposition 3, there exist infinitely many k-critical digraphs for which

$$\frac{||D||}{|D|} \leq \frac{e-1}{v-1} \leq \frac{v^2-v-2}{2(v-1)} = \frac{v}{2} - \frac{1}{v-1} < \frac{v}{2} \leq \frac{k^2-k+1}{2}.$$

This proves Theorem 2.

6. Conclusion

This research was motivated by the edge density question for digraphs, i.e., whether there exist functions $x(k)$ and $y(k)$ such that for every $k \geq 3$ there are infinitely many dense k-critical digraphs satisfying $||D|| > x(k)|D|^2$, and infinitely many sparse k-critical digraphs satisfying $||D|| < y(k)|D|$. By Theorems 1 and 2, $x(k) = \left(\frac{1}{2} - \frac{1}{2^{k-1}}\right)$ and $y(k) = \frac{k^2-k+1}{2}$ are two such functions.

Naturally, this motivates the question of whether these functions $x(k)$ and $y(k)$ are optimal. For example, we know that $y(3) = \frac{7}{2}$ is not optimal, as we showed in Section 5 that this fraction could be $\frac{19}{6} < \frac{7}{2}$. But can we do even better?

The answer is yes; Figure 3 presents a 3-critical digraph D with $|D| = 8$ and $||D|| = 21$, which by Proposition 3 implies the existence of infinitely many 3-critical digraphs with $||D|| < \frac{20}{7}|D|$.

This analysis motivates the following open problem.

Problem 1. *For each $k \geq 3$, determine a function $y^*(k) < y(k)$ for which there exist infinitely many k-critical digon-free digraphs satisfying $||D|| < y^*(k)|D|$. What is the infimum of all such functions $y^*(k)$?*

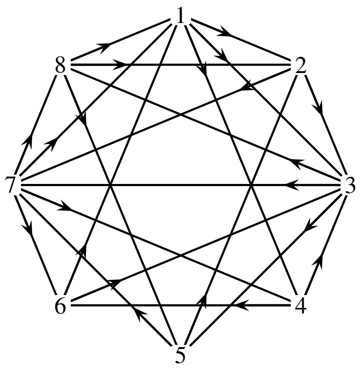

Figure 3. A 3-critical digraph on 8 vertices and 21 edges.

We believe that the optimal function satisfies $y^*(k) = \frac{k^2}{2} - O(k)$, but do not have a formal proof of this conjecture.

As for the upper bound, we conjecture that the function $x(k) = \left(\frac{1}{2} - \frac{1}{2^{k-1}}\right)$ is optimal, or close-to-optimal. Note that as $k \to \infty$, this function approaches $\frac{1}{2}$ which is asymptotically best possible. We conclude with one more open problem.

Problem 2. *For each $k \geq 3$, determine a function $x^*(k) > x(k)$ for which there exist infinitely many k-critical digon-free digraphs satisfying $||D|| > x^*(k)|D|^2$. What is the supremum of all such functions $x^*(k)$?*

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References

[1] G. Araujo-Pardo and M. Olsen: A conjecture of Neumann-Lara on infinite families of r-dichromatic circulant tournaments, *Discrete Mathematics* **310** (2010), 489–492.

[2] E. Berger, K. Choromanski, M. Chudnovsky, J. Fox, M. Loebl, A. Scott, P. Seymour and S. Thomassé: Tournaments and colouring, *Journal of Combinatorial Theory Ser. B* **103** (2013), 1–20.

[3] D. Bokal, G. Fijavz, M. Juvan, P. Kayll M. and B. Mohar: The circular chromatic number of a digraph, *Journal of Graph Theory* **46** (2004), 227–240.

[4] B. Bollobás: Chromatic number, girth, and maximal degree, *Discrete Mathematics* **24** (1978), 311–314.

[5] G. A. Dirac: A property of 4-chromatic graphs and some remarks on critical graphs, *Journal of the London Mathematical Society* **27** (1952), 85–92.
[6] P. Erdős and A. Hajnal: Ramsey-type theorems, *Discrete Applied Mathematics* **25** (1989), 37–52.
[7] B. Farzad and M. Molloy: On the edge density of 4-critical graphs, *Combinatorica* **29** (2009), 665–689.
[8] T. Gallai: Kritische Graphen I, *Publ. Math. Inst. Hungar. Acad. Sci.* **8** (1963), 165–192.
[9] A. Harutyunyan and B. Mohar: Gallai's theorem for list coloring of digraphs, *SIAM Journal on Discrete Mathematics* **25** (2011), 170–180.
[10] A. Harutyunyan and B. Mohar: Two results on the digraph chromatic number, *Discrete Mathematics* **312** (2012), 1823–1826.
[11] T. R. Jensen: Dense critical and vertex-critical graphs, *Discrete Mathematics* **258** (2002), 63–84.
[12] T. R. Jensen and B. Toft: *Graph coloring problems,* New York, Wiley Interscience, 1995.
[13] A. V. Kostochka and M. Stiebitz: A new lower bound on the number of edges in colour-critical graphs and hypergraphs, *Journal of Combinatorial Theory Series B* **87** (2003), 374–402.
[14] A. V. Kostochka and M. Yancey: Ore's Conjecture on color-critical graphs is almost true, to appear.
[15] V. Neumann-Lara: The dichromatic number of a digraph, *Journal of Combinatorial Theory Series B* **33** (1982), 265–270.
[16] V. Neumann-Lara: The 3 and 4-dichromatic tournaments of minimum order, *Discrete Mathematics* **135** (1994), 233–243.
[17] B. Mohar: Circular colorings of edge-weighted graphs, *Journal of Graph Theory* **43** (2003), 107–116.
[18] B. Mohar: Eigenvalues and colorings of graphs, *Linear Algebra and its Applications* **432** (2010), 2273–2277.
[19] B. Toft: On the maximal number of edges of critical k-chromatic graphs, *Studia Scientiarum Mathematicarum Hungarica* **5** (1970), 461–470.
[20] P. Turán: On an extremal problem in graph theory, *Mat. Fiz. Lapok* **48** (1941), 436–452.
[21] H. S. Wilf: The eigenvalues of a graph and its chromatic number, *Journal of the London Mathematical Society* **42** (1967), 330–332.

Richard Hoshino

Quest University Canada
Squamish, British Columbia
`richard.hoshino@gmail.com`

Ken-ichi Kawarabayashi

National Institute of Informatics
Tokyo,
JST-ERATO Kawarabayashi
Large Graph Project
`k_keniti@nii.ac.jp`