Proof of a Conjecture on Fractional Ramsey Numbers

Jason Brown and Richard Hoshino

DEPARTMENT OF MATHEMATICS AND STATISTICS
DALHOUSIE UNIVERSITY HALIFAX
NOVA SCOTIA, CANADA B3H 3J5
E-mail: brown@mathstat.dal.ca; richard.hoshino@gmail.com

Received July 15, 2007; Revised March 11, 2009

Published online 9 April 2009 in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/jgt.20416

Abstract: Jacobson, Levin, and Scheinerman introduced the fractional Ramsey function $r_f(a_1, a_2, \ldots, a_k)$ as an extension of the classical definition for Ramsey numbers. They determined an exact formula for the fractional Ramsey function for the case $k=2$. In this article, we answer an open problem by determining an explicit formula for the general case $k>2$ by constructing an infinite family of circulant graphs for which the independence numbers can be computed explicitly. This construction gives us two further results: a new (infinite) family of star extremal graphs which are a superset of many of the families currently known in the literature, and a broad generalization of known results on the chromatic number of integer distance graphs. © 2009 Wiley Periodicals, Inc. J Graph Theory 63: 164–178, 2010

Keywords: *Ramsey theory; circulant graph; independence number; star extremal; chromatic number; fractional chromatic number; circular chromatic number; integer distance graph*

Contract grant sponsor: Natural Sciences and Engineering Council.

© 2009 Wiley Periodicals, Inc.

1. INTRODUCTION

For any graph G, we define $\alpha(G)$ to be its *independence number*, the maximum number of vertices that can be selected so that no two vertices are adjacent. As with many other graph parameters, there is no known polynomial-time algorithm to compute $\alpha(G)$. Computing $\alpha(G)$ is NP-hard (c.f. [13]).

Circulants are symmetric graphs, and are a subset of the more general family of Cayley graphs. Each circulant graph is characterized by its order n, and its *generating set* $S \subseteq \{1,2,\dots,\lfloor \frac{n}{2} \rfloor\}$. (In the literature, the generating set is also known as the *connection set*.) We define $G = C_{n,S}$ to be the graph with vertex set $V(G) = \mathbb{Z}_n$ and edge set $E(G) = \{uv : |u-v|_n \in S\}$, where $|x|_n = \min\{|x|, n-|x|\}$ is the *circular distance* modulo n.

Owing to the structure and symmetry of a circulant graph, one might expect that there is a simple algorithm to compute $\alpha(G)$ when G is restricted to circulants. However, in [5], it is proven that determining the independence number of an arbitrary circulant graph is NP-hard. Moreover, it is shown [5] that it is NP-hard even to get a good approximation for $\alpha(G)$. A formula for $\alpha(C_{n,S})$ is known only for a handful of generating sets.

We will construct an infinite family of circulant graphs for which the independence number $\alpha(G)$ can be determined explicitly. In Section 2, we provide necessary definitions on circulant graphs and star-extremal graphs to state two key theorems, which are then proven in Section 3. In Section 4, we apply the results of Section 3 to determine an explicit formula for the fractional Ramsey function, solving an open problem of Jacobson, Levin, and Scheinerman [12, 16]. We conclude the article by applying these results to generalize known results on the chromatic number of integer distance graphs.

2. DEFINITIONS

Let $S \subseteq \{1,2,\dots,\lfloor \frac{n}{2} \rfloor\}$ be the generating set for some circulant $G = C_{n,S}$, with $|V(G)| = n$. Define

$$\pm S \pmod{n} = \{x : x \in S \text{ or } n-x \in S\}.$$

Fix a k-tuple of integers $(a_1, a_2, \dots, a_k)$ with each $a_i \geq 3$. Given our k-tuple, we construct the graphs $G_{j,k} = G_j(a_1, a_2, \dots, a_k)$ as follows.

Define $n_0 = 1$, and $n_i = a_i n_{i-1} - 1$, for $1 \leq i \leq k$.

For each $1 \leq j \leq i \leq k$, define

$$S_{j,i} = \begin{cases} \pm S_{j,i-1} \pmod{n_{i-1}} & \text{for all } 1 \leq j < i \\ \left\{1,2,\dots,\left\lfloor \frac{n_i}{2} \right\rfloor\right\} - \bigcup_{t=1}^{i-1} S_{t,i} & \text{for } j = i \end{cases}$$

Define $G_{j,k}$ to be the circulant graph $C_{n_k, S_{j,k}}$ on n_k vertices with generating set $S_{j,k}$.

To illustrate our construction, let $(a_1, a_2, a_3) = (5,6,8)$. Then, we derive $(n_1, n_2, n_3) = (4, 23, 183)$. We have $S_{1,1} = \{1,2\}$, $S_{1,2} = \{1,2,3\}$, $S_{2,2} = \{4,5,6,\dots,11\}$,

$S_{1,3}=\{1,2,3,20,21,22\}$, $S_{2,3}=\{4,5,\ldots,19\}$, and $S_{3,3}=\{23,24,\ldots,91\}$. Thus, $G_{1,3}=C_{183,\{1,2,3,20,21,22\}}$, $G_{2,3}=C_{183,\{4,5,\ldots,19\}}$, and $G_{3,3}=C_{183,\{23,24,\ldots,91\}}$.

Note that in our example, the $S_{j,k}$'s form a partition of $\{1,2,\ldots,\lfloor\frac{n_k}{2}\rfloor\}$. It is a straightforward induction to show that this is always the case, and so we omit the details.

Lemma 2.1. *The $S_{j,k}$'s form a partition of $\{1,2,\ldots,\lfloor\frac{n_k}{2}\rfloor\}$. In other words, the $G_{j,k}$'s form an edge partition of the complete graph K_{n_k}.*

We now give two definitions.

Definition 2.2. For any generating set $S=\{s_1,s_2,\ldots,s_m\}$ with $1\le s_1<s_2<\cdots<s_m\le\lfloor\frac{n}{2}\rfloor$, define its **end sum** to be $\Omega(S)=\min(S)+\max(S)=s_1+s_m$. Say that S is **reversible** if $s_i+s_{m+1-i}=\Omega(S)$ for all $1\le i\le m$.

In our example, $S_{1,3}$ and $S_{2,3}$ are both reversible sets with end sum $n_2=23$. We make several observations, which will be useful later in the article. As all of these results follow immediately from the definitions, the proofs are omitted.

Proposition 2.3. *For any k-tuple of positive integers $(a_1,a_2,\ldots,a_k)$ with each $a_i\ge 3$, the $S_{j,k}$'s satisfy the following conditions:*

(a) $S_{j,k}=S_{j,k-1}\cup\{n_{k-1}-x:x\in S_{j,k-1}\}$, *for all* $1\le j\le k-1$.
(b) $S_{j,i-1}\subset S_{j,i}$ *for each* $1\le j<i\le k$.
(c) $0\le n_{k-1}-1<\lfloor\frac{n_k}{2}\rfloor$, *with equality iff* $k=1$.
(d) $S_{k,k}=\{n_{k-1},n_{k-1}+1,\ldots,\lfloor\frac{n_k}{2}\rfloor\}$.
(e) $S_{k-1,k}=\{n_{k-2},n_{k-2}+1,\ldots,n_{k-1}-n_{k-2}\}$.
(f) *For every* $1\le j\le k$, *the generating set* $S_{j,k}$ *is reversible. Furthermore, if* $1\le j\le k-1$, *then* $\Omega(S_{j,k})=n_{k-1}$.
(g) *For all* $1\le j\le k-1$, $\max(S_{j,k})\le n_{k-1}-1$.

We say that an *interval* of a generating set $S_{j,k}$ is a maximum sequence of consecutive terms. For example, $S_{1,3}=\{1,2,3,20,21,22\}$ consists of two intervals of length 3, namely the interval $\{1,2,3\}$ and the interval $\{20,21,22\}$.

As mentioned in the introduction, a formula for $\alpha(C_{n,S})$ is known only for a handful of generating sets S. The majority of these known sets S have only one or two intervals. Our first key theorem will prove an exact formula for $\alpha(G_{j,k})$, for every $1\le j\le k$. This is a significant extension of previously published results, as the following lemma shows that the generating sets $S_{j,k}$ can have *arbitrarily* many intervals of equal length.

Lemma 2.4. *Let $1\le j\le k-1$. Then $S_{j,k}$ contains 2^{k-j-1} intervals of equal length.*

Proof. We proceed by induction on k. The base case $k=2$ is trivial, as $S_{1,2}=\{1,2,\ldots,n_1-1\}$. Let $k\ge 3$ and suppose the lemma is true for $k-1$. Then, for all $1\le j\le k-2$, the set $S_{j,k-1}$ contains 2^{k-j-2} intervals of equal length. Let $S_{j,k-1}=\{s_1,s_2,\ldots,s_t\}$, where $s_1<s_2<\cdots<s_t$. By Proposition 2.3(f), $S_{j,k}=\{s_1,s_2,\ldots,s_t,s_{t+1},s_{t+2},\ldots,s_{2t}\}$ where $s_i+s_{2t+1-i}=n_{k-1}$ for all $1\le i\le 2t$. Each half of $S_{j,k}$ (namely the first t elements and the last t elements) contains exactly 2^{k-j-2} intervals of equal length, by the

induction hypothesis. Now we show that no elements overlap, and that these intervals are disjoint. By Proposition 2.3(g), $\max(S_{j,k-1}) = s_t \leq n_{k-2}-1$, and so

$$\begin{aligned} s_{t+1}-s_t &= (n_{k-1}-s_t)-s_t \\ &= n_{k-1}-2s_t \\ &\geq n_{k-1}-2(n_{k-2}-1) \\ &= (a_{k-1}n_{k-2}-1)-2n_{k-2}+2 \\ &= (a_{k-1}-2)n_{k-2}+1 \\ &> 1. \end{aligned}$$

It follows that all of these intervals are disjoint. Thus, $S_{j,k}$ contains 2^{k-j-1} intervals of equal length, completing the induction. Finally, if $j=k-1$, then we need to show that $S_{k-1,k}$ consists of just $2^{k-j-1}=1$ interval. This follows as $S_{k-1,k}=\{n_{k-2}, n_{k-2}+1, \ldots, n_{k-1}-n_{k-2}\}$, by Proposition 2.3(e). ■

We have now established that $S_{j,k}$ can have arbitrarily many intervals. Thus, finding a formula for $\alpha(G_{j,k})$ will establish a new class of graphs for which the independence number can be computed explicitly. This motivates the definition of star-extremal graphs, which have an important connection to independence numbers.

A graph G is *star extremal* if its fractional chromatic number $\chi_f(G)$ equals its circular chromatic number $\chi_c(G)$. Let us define these two graph parameters.

The *chromatic number* of a graph, $\chi(G)$, is the smallest size of a cover of the vertices of G by independent sets. We can alternately define $\chi(G)$ using an *integer program* (IP). For more information on linear and IPs, we refer the reader to [4].

Let M denote the vertex-independent set incidence matrix of G. The rows are indexed by the vertices $\{v_1, v_2, \ldots, v_n\}$, and the columns are indexed by the independent subsets of the vertices, $\{I_1, I_2, \ldots, I_m\}$. The (i,j) entry of M is 1 when $v_i \in I_j$, and is 0 otherwise. Then, $\chi(G) = \min \mathbf{1}\cdot\mathbf{x}$, where $M\mathbf{x} \geq \mathbf{1}$, $\mathbf{x} \geq \mathbf{0}$, and $\mathbf{x} \in \mathbb{Z}^m$. Here, $\mathbf{1}$ denotes a vector of all 1's.

Definition 2.5 (Hell and Nesetril [9], Scheinerman and Ullman [21]). The **fractional chromatic number** $\chi_f(G)$ is the relaxation of the IP of $\chi(G)$ into the linear program:

$$\chi_f(G) = \min \mathbf{1}\cdot\mathbf{x}, \quad \text{where } M\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}, \text{ and } \mathbf{x} \in \mathbb{R}^m.$$

By taking the IP of a graph parameter and relaxing the IP into the linear program, we may define a corresponding *fractional* analogue. For more information on the uses and applications of fractional graph theory, we refer the reader to [9, 21].

Now we define the circular chromatic number $\chi_c(G)$. Let k and d be positive integers such that $k \geq 2d$. A (k,d)-*coloring* of a graph $G=(V,E)$ is a mapping $C: V \to \{0, 1, \ldots, k-1\}$ such that the circular distance satisfies the inequality $|C(x)-C(y)|_k \geq d$ for any $xy \in E(G)$.

Definition 2.6 (Vince [22]). The **circular chromatic number** $\chi_c(G)$ is the infimum of $\frac{k}{d}$ for which there exists a (k,d)-coloring of G.

Notice that $\chi(G)$ is just the smallest k for which there exists a $(k,1)$-coloring of G. Thus, $\chi_c(G)$ is a generalization of $\chi(G)$. The circular chromatic number is sometimes referred to as the *star chromatic number* [22, 24]. In [22], the following result is proven.

Theorem 2.7 (Vince [22]). *For any graph G, $\chi(G) = \lceil \chi_c(G) \rceil$.*

Therefore, knowing $\chi_c(G)$ immediately determines the chromatic number. The following set of inequalities is known.

Lemma 2.8 (Lih et al. [17]).

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leq \chi_f(G) \leq \chi_c(G) \leq \chi(G).$$

Since $\chi_f(G) \leq \chi_c(G)$ for all G, a natural question is to investigate when these two parameters are equal. This motivates the following definition.

Definition 2.9 (Gao and Zhu [8]). A graph G is *star extremal* if $\chi_f(G) = \chi_c(G)$.

The notion of star extremality in graphs was first introduced in the study of the chromatic number and the circular chromatic number of the lexicographic product of graphs. It is shown [8] that $\chi_c(G[H]) = \chi_c(G)\chi(H)$ for all star-extremal graphs G. (Note that $G[H]$ refers to the lexicographic product of G with H, arising from replacing each vertex of G with a copy of H.) In other words, if G is star extremal, one can immediately compute the circular chromatic number of $G[H]$, and hence, the chromatic number of $G[H]$.

In [8], the following *multiplier method* is introduced, which provides a sufficient condition for a circulant graph G to be star extremal.

Given $G = C_{n,S}$, and a positive integer t, let

$$\lambda_t(G) = \min\{|ti|_n : i \in S\}.$$

Then define

$$\lambda(G) = \max\{\lambda_t(G) : t = 1, 2, 3, \ldots, n\}.$$

We note that, unlike $\alpha(G)$, the parameter $\lambda(G)$ can be determined in polynomial-time for any $G = C_{n,S}$. The following lemma provides a sufficient condition for a circulant graph to be star extremal.

Lemma 2.10 (Gao and Zhu [8]). *Let $G = C_{n,S}$. Then, $\lambda(G) \leq \alpha(G)$. Furthermore, if $\lambda(G) = \alpha(G)$, then $\chi_f(G) = \chi_c(G) = \frac{n}{\alpha(G)}$, i.e., G is star extremal.*

In the following section, we will show that the exact value of $\alpha(G) = \alpha(C_{n,S})$ can be determined for every generating set S in our infinite family $S_{j,k}$. Thus, it is a natural question to consider the star extremality of these graphs. If we can calculate a formula for $\lambda(G)$ for every $G = G_{j,k}$ in our infinite family, we can compare it to $\alpha(G)$ and check if these values are equal. This is how we will show that *every* $G_{j,k}$ is star extremal.

It turns out that many of the star extremal circulant graphs currently known in the literature [1, 11, 15, 17, 18] are specific examples of circulants in our infinite family. Thus, our construction is a broad generalization of previously published results.

3. A FAMILY OF STAR-EXTREMAL CIRCULANTS

For a general circulant graph $C_{n,S}$, it is NP-hard [5] to compute the value of $\alpha(C_{n,S})$. However, the following theorem shows that for every k-tuple satisfying a given sequence of inequalities, we can compute the exact value of $\alpha(C_{n,S})=\alpha(G_{j,k})$, for all $1\le j\le k$.

Theorem 3.1. *Let $a_1,a_2,\dots,a_k$ be integers such that $a_k\ge a_{k-1}\ge\cdots\ge a_1\ge 3$. Consider the circulant graph $G_{j,k}=G_j(a_1,a_2,\dots,a_k)$. Then*

$$\alpha(G_{j,k})=\begin{cases} a_k\alpha(G_{j,k-1})-1 & \text{for } 1\le j\le k-1\\ n_{k-1} & \text{for } j=k\end{cases}$$

This theorem allows us to determine $\alpha(G_{j,k})$, for any k-tuple $(a_1,a_2,\dots,a_k)$ satisfying the given inequality. This powerful theorem enables us to compute exact values of $\alpha(G)$ for an infinite family of circulant graphs, rather than just upper and lower bounds. We then show that each $G_{j,k}$ is star extremal:

Theorem 3.2. *Let $a_1,a_2,\dots,a_k$ be integers such that $a_k\ge a_{k-1}\ge\cdots\ge a_1\ge 3$. Then $G_{j,k}=G_j(a_1,a_2,\dots,a_k)$ is star extremal for all $1\le j\le k$.*

Note that Theorem 3.1 gives us a simple linear-time recursive method to determine $\alpha(G)$ for any graph in this infinite family. To illustrate, we compute the independence number of $C_{183,\{1,2,3,20,21,22\}}$. Recall that this graph is $G_{1,3}=G_1(a_1,a_2,a_3)$, where $(a_1,a_2,a_3)=(5,6,8)$, $j=1$ and $k=3$.

$$\begin{aligned}\alpha(C_{183,\{1,2,3,20,21,22\}}) &= 8\alpha(C_{23,\{1,2,3\}})-1\\ &= 8(6\alpha(C_{4,\{1,2\}})-1)-1\\ &= 8(6\cdot 1-1)-1\\ &= 39.\end{aligned}$$

We first prove the two theorems for the $j=k$ case. Then we use induction to simultaneously establish the two theorems for the case $1\le j\le k-1$.

First, we prove the $j=k$ case directly.

Proposition 3.3. *Let $k\ge 1$. Then, $\alpha(G_{k,k})=n_{k-1}$. Furthermore, $G_{k,k}$ is star extremal.*

Proof. By Proposition 2.3(d), $S_{k,k}=\{n_{k-1},n_{k-1}+1,\dots,\lfloor\frac{n_k}{2}\rfloor\}$. To show that $\alpha(G_{k,k})\ge n_{k-1}$, it suffices to find an independent set I of $G_{k,k}$ with n_{k-1} vertices. The set $I=\{0,1,2,\dots,n_{k-1}-1\}$ satisfies this property. But if I is an independent set with $|I|>n_{k-1}$, then there must be two elements with a circular distance of at least n_{k-1}, a contradiction. Hence, $\alpha(G_{k,k})=n_{k-1}$.

We see that $\lambda_t(G_{k,k})=n_{k-1}$ when $t=1$, since $S_{k,k}=\{n_{k-1},n_{k-1}+1,\ldots,\lfloor\frac{n_k}{2}\rfloor\}$. By definition, $\lambda(G_{k,k})\geq\lambda_1(G_{k,k})=n_{k-1}$. From Lemma 2.10 and the previous paragraph, we have $n_{k-1}\leq\lambda(G_{k,k})\leq\alpha(G_{k,k})=n_{k-1}$. This implies that $\lambda(G_{k,k})=\alpha(G_{k,k})$, i.e., $G_{k,k}$ is star extremal. ■

This establishes the two theorems for the case $j=k$. Now we tackle the case $1\leq j\leq k-1$. First, we prove that $\alpha(G_{j,k})\leq a_k\alpha(G_{j,k-1})-1$, for each $1\leq j\leq k-1$. The result will follow immediately from two lemmas. The first lemma is a result of Collins [6].

Lemma 3.4 (Collins [6]). *Let $G=C_{n,S}$. Let G_S be the subgraph of G induced by taking any set of $\Omega(S)$ consecutive vertices in G. Then, $\frac{\alpha(G)}{|G|}\leq\frac{\alpha(G_S)}{|G_S|}$.*

Lemma 3.5. *Let $1\leq j\leq k-1$ be fixed. Let H be the subgraph of $G_{j,k}$ induced by taking any set of $\Omega(S_{j,k})$ consecutive vertices in $G_{j,k}$. Then, $H\simeq G_{j,k-1}$.*

Proof. By Proposition 2.3(f), $S_{j,k}$ is reversible with $\Omega(S_{j,k})=n_{k-1}$. Since $G_{j,k}$ is a circulant, all induced subgraphs of $G_{j,k}$ with n_{k-1} consecutive vertices are isomorphic. So without loss, we can take H to be the subgraph of $G_{j,k}$ induced by the vertices $0,1,2,\ldots,n_{k-1}-1$. To show $H\simeq G_{j,k-1}$, we prove that $uv\in E(H)$ iff $uv\in E(G_{j,k-1})$. The latter is equivalent to the condition $|u-v|_{n_{k-1}}\in S_{j,k-1}$.

Let $0\leq u<v\leq n_{k-1}-1$. Since $n_k=a_kn_{k-1}-1>2(n_{k-1}-1)$, $uv\in E(H)$ iff $|u-v|_{n_k}=v-u\in S_{j,k}$. By Proposition 2.3(a), $S_{j,k}=S_{j,k-1}\cup\{n_{k-1}-x:x\in S_{j,k-1}\}$, and so $v-u\in S_{j,k}$ iff $v-u\in S_{j,k-1}$ or $n_{k-1}-(v-u)\in S_{j,k-1}$. In other words, $uv\in E(H)$ iff $|u-v|_{n_{k-1}}\in S_{j,k-1}$. This proves that $H\simeq G_{j,k-1}$, and our proof is complete. ■

Proposition 3.6. *For each $1\leq j\leq k-1$, we have $\alpha(G_{j,k})\leq a_k\alpha(G_{j,k-1})-1$.*

Proof. From Lemmas 3.4 and 3.5, $\frac{\alpha(G_{j,k})}{|G_{j,k}|}\leq\frac{\alpha(G_{j,k-1})}{|G_{j,k-1}|}$. Since $|G_{j,k}|=n_k$ and $|G_{j,k-1}|=n_{k-1}$, we have

$$\alpha(G_{j,k})\leq\frac{n_k}{n_{k-1}}\alpha(G_{j,k-1})=\frac{a_kn_{k-1}-1}{n_{k-1}}\alpha(G_{j,k-1})<a_k\alpha(G_{j,k-1}).$$

Since $\alpha(G_{j,k})<a_k\alpha(G_{j,k-1})$, this implies that $\alpha(G_{j,k})\leq a_k\alpha(G_{j,k-1})-1$, as required. ■

As discussed in the previous section, Lemma 2.10 is the main technique used to verify that a circulant $G=C_{n,S}$ is star extremal. For each $1\leq j\leq k-1$, we will find a t for which $\lambda_t(G_{j,k})=\alpha(G_{j,k})$. By Lemma 2.10, this will imply that $\alpha(G_{j,k})=\lambda_t(G_{j,k})\leq\lambda(G_{j,k})\leq\alpha(G_{j,k})$, i.e., $\lambda(G_{j,k})=\alpha(G_{j,k})$. In other words, to prove that $G_{j,k}$ is star extremal, it suffices to find *one* multiplier t for which $\lambda_t(G_{j,k})=\alpha(G_{j,k})$.

The following theorem tells us how to find this multiplier t. The proof of this result is long and complicated, and appears in the second author's Ph.D thesis.

Theorem 3.7 (Hoshino [10]). *Let j and k be fixed integers satisfying $1\leq j\leq k-1$. Let $a_1,a_2,\ldots,a_k$ be integers such that $a_k\geq a_{k-1}\geq\cdots\geq a_1\geq 3$. Define $t_j=1$ and $t_k=a_{j+1}a_{j+2}\ldots a_k$ for all $k\geq j+1$. Then $t=t_k$ is the desired multiplier satisfying $\lambda_t(G_{j,k})=\alpha(G_{j,k})$. Furthermore, $\lambda_{t_k}(G_{j,k})=a_k\lambda_{t_{k-1}}(G_{j,k-1})-1$.*

We are now ready to prove our two main results: Theorem 3.1 and 3.2. Recall that the case $j=k$ has already been established. So it suffices to prove the theorems for j and k satisfying $1 \le j \le k-1$. We do this by induction on k. In our inductive proof, we prove both theorems simultaneously.

Proof. We proceed by induction on k. First, we establish the base case $k=2$. If $k=2$, then j must be 1. Then $t_k = t_2 = a_2$. Thus, to verify the base case, we must prove that $\alpha(G_{1,2}) = a_2\alpha(G_{1,1}) - 1$ and that $\alpha(G_{1,2}) = \lambda(G_{1,2}) = \lambda_{a_2}(G_{1,2})$.

Note that $n_0 = 1$ and $n_1 = a_1 - 1$. By Proposition 2.3(d), $S_{1,2} = \{1, 2, \ldots, a_1 - 2\}$. Thus, $G_{1,2} = C_{n_2, S_{1,2}}$, where $n_2 = a_2 n_1 - 1 = a_2(a_1 - 1) - 1$.

We see that $I = \{0, a_1 - 1, 2(a_1 - 1), \ldots, (a_2 - 2)(a_1 - 1)\}$ is an independent set of $G_{1,2}$. Thus, $\alpha(G_{1,2}) \ge |I| = a_2 - 1$. Suppose that $\alpha(G_{1,2}) \ge a_2$. Then there exist vertices $v_1, v_2, \ldots, v_{a_2}$ that are independent in $G_{1,2}$. In other words, the distance between any pair of these vertices must be at least $a_1 - 1$. This implies that $|G| = n_2 \ge (a_1 - 1)a_2$, which is a contradiction since $n_2 = a_2 n_1 - 1 = a_2(a_1 - 1) - 1$. Hence, $\alpha(G_{1,2}) = a_2 - 1$. Noting that $G_{1,1}$ is the complete graph, we have $\alpha(G_{1,1}) = 1$ and so $\alpha(G_{1,2}) = a_2\alpha(G_{1,1}) - 1$.

Let $t = a_2$. Then $\lambda_{a_2}(G_{1,2})$ is the minimum value of $|a_2 i|_{n_2}$ over all $i \in S_{1,2}$. It is easy to see that this minimum is achieved when $i = a_1 - 2$ and so $\lambda_{a_2}(G_{1,2}) = |a_2(a_1 - 2)|_{n_2} = a_2 - 1$. By definition, $\lambda(G_{1,2}) \ge \lambda_{a_2}(G_{1,2}) = a_2 - 1$. From Lemma 2.10 and the previous paragraph, we have $a_2 - 1 \le \lambda(G_{1,2}) \le \alpha(G_{1,2}) = a_2 - 1$. This implies that $\alpha(G_{1,2}) = \lambda(G_{1,2}) = \lambda_{a_2}(G_{1,2})$, i.e., $G_{1,2}$ is star extremal.

Having established the base case, we now complete the proof by induction. Specifically, suppose that $\alpha(G_{j,k-1}) = \lambda(G_{j,k-1}) = \lambda_{t_{k-1}}(G_{j,k-1})$. (Surprisingly, we will only need the induction hypothesis for one of the two theorems.) Then, we have

$$
\begin{aligned}
a_k\lambda(G_{j,k-1}) - 1 &= a_k\lambda_{t_{k-1}}(G_{j,k-1}) - 1 && \text{by the induction hypothesis} \\
&= \lambda_{t_k}(G_{j,k}) && \text{by Theorem 3.7} \\
&\le \lambda(G_{j,k}) && \text{by definition} \\
&\le \alpha(G_{j,k}) && \text{by Lemma 2.10} \\
&\le a_k\alpha(G_{j,k-1}) - 1 && \text{by Proposition 3.6} \\
&= a_k\lambda(G_{j,k-1}) - 1 && \text{by the induction hypothesis.}
\end{aligned}
$$

Therefore, we have equality throughout. We have $\alpha(G_{j,k}) = \lambda(G_{j,k}) = \lambda_{t_k}(G_{j,k})$, proving the star extremality of $G_{j,k}$. As an added bonus, this chain of inequalities shows that $\alpha(G_{j,k}) = a_k\alpha(G_{j,k-1}) - 1$.

By induction, we have established Theorems 3.1 and 3.2 for $1 \le j \le k-1$. Since these two theorems have already been established for $j = k$, our proof is complete. ■

4. THE FRACTIONAL RAMSEY THEORY FUNCTION

We now apply the results of the previous section to find an explicit formula for the *fractional* Ramsey function, an analogue to the well-known Ramsey function. Before we define this function, we first give an alternative definition for the clique number $\omega(G)$. It is well-known [21] that $\omega(G)$ is the dual of the IP for $\chi(G)$. Therefore, $\omega(G) = \max \mathbf{1}\cdot\mathbf{y}$, where $M^t\mathbf{y} \leq \mathbf{1}$, $\mathbf{y} \geq \mathbf{0}$, and $\mathbf{y} \in \mathbb{Z}^n$. (Recall that M denotes the vertex-independent set incidence matrix of G.)

Definition 4.1. Let $a_1, a_2, \ldots, a_k$ be positive integers. Then the (classical) **Ramsey number** $r(a_1, a_2, \ldots, a_k)$ is the smallest positive integer n such that if $H_1, H_2, \ldots, H_k$ are any graphs for which

$$K_n = H_1 \oplus H_2 \oplus \cdots \oplus H_k,$$

then for some $1 \leq i \leq k$,

$$\max_{\mathbf{y}\in\mathbb{Z}^n, M_i^t\mathbf{y}\leq\mathbf{1}} \mathbf{1}\cdot\mathbf{y} = \omega(H_i) \geq a_i.$$

Thus, if the H_i's represent an edge decomposition of K_n, then at least one index i satisfies the inequality $\omega(H_i) \geq a_i$.

Now we define the *fractional Ramsey number* $r_f(a_1, a_2, \ldots, a_k)$, which is introduced in [12, 16]. This is the same definition as $r(a_1, a_2, \ldots, a_k)$, except the clique number $\omega(G)$ is replaced by the *fractional clique number* $\omega_f(G)$, where we relax the IP for $\omega(G)$ into the linear program.

Definition 4.2. Let $a_1, a_2, \ldots, a_k$ be positive integers. Then the **fractional Ramsey number** $r(a_1, a_2, \ldots, a_k)$ is the smallest positive integer n such that if $H_1, H_2, \ldots, H_k$ are any graphs for which

$$K_n = H_1 \oplus H_2 \oplus \cdots \oplus H_k,$$

then for some $1 \leq i \leq k$,

$$\max_{\mathbf{y}\in\mathbb{R}^n, M_i^t\mathbf{y}\leq\mathbf{1}} \mathbf{1}\cdot\mathbf{y} = \omega_f(H_i) \geq a_i.$$

By the duality theorem of linear programming [4], $\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G)$, for any graph G.

Ramsey's Theorem [20] states that $r(a_1, a_2, \ldots, a_k)$ is well-defined for each choice of the a_i's. This implies that $r_f(a_1, a_2, \ldots, a_k)$ must also exist for any choice of the a_i's, as it is bounded above by $r(a_1, a_2, \ldots, a_k)$, since $\omega(H_i) \leq \omega_f(H_i)$, for each subgraph H_i. Despite much effort into computing, the exact values of Ramsey numbers, no one has made any progress in determining an explicit formula for the Ramsey function, even for the case $k = 2$.

The fractional Ramsey function $r_f(a_1, a_2, \ldots, a_k)$ was introduced in [12, 16]. As with the classical function, the fractional Ramsey function has the following basic properties, where each a_i is a positive integer:

(a) If any $a_i = 1$, then $r_f(a_1, a_2, \ldots, a_k) = 1$.
(b) $r_f(a_1) = a_1$ and $r_f(2, a_2, a_3, \ldots, a_k) = r_f(a_2, a_3, \ldots, a_k)$.
(c) The function r_f is invariant under permutation of the a_i's.

Therefore, we may assume without loss that $k \geq 2$ and that the a_i's satisfy the inequality $a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 3$.

An explicit formula for $r_f(a_1, a_2)$ is found in [12, 16], and a formula for $r_f(a_1, a_2, \ldots, a_k)$ is left as an open problem. We now determine this formula for any k-tuple $(a_1, a_2, \ldots, a_k)$ of positive integers, by establishing the following theorem. The proof is a simple corollary from the results of the previous section.

Theorem 4.3. *Let $a_1, a_2, \ldots, a_k$ be integers such that $a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 3$. Then*

$$r_f(a_1, a_2, \ldots, a_k) = \prod_{j=1}^{k} a_j - \sum_{i=2}^{k} \left(\prod_{j=i}^{k} a_j \right).$$

Recall that in Section 2, we defined $n_0 = 1$ and $n_i = a_i n_{i-1} - 1$ for each $1 \leq i \leq k$. It is a straightforward induction to verify that the right side of the identity in Theorem 4.3 is equal to $n_k + 1$. Thus, it suffices to prove that $r_f(a_1, a_2, \ldots, a_k) = n_k + 1$.

The upper bound $r_f(a_1, a_2, \ldots, a_k) \leq n_k + 1$ is proven in [12], and is just a simple application of the following lemma (note the similarity of this lemma to Lemma 2.10).

Lemma 4.4 (Jacobson et al. [12]). *Let G be a graph on n vertices. Then $\omega_f(G) \geq \frac{n}{\alpha(G)}$. In the case that G is vertex-transitive, $\omega_f(G) = \frac{n}{\alpha(G)}$.*

The difficulty lies in proving the lower bound $r_f(a_1, a_2, \ldots, a_k) > n_k$, a result conjectured in [12, 16]. From the analysis earlier in this article, the proof is straightforward. We just require one lemma, and then we can establish Theorem 4.3.

Lemma 4.5. *Let $a_1, a_2, \ldots, a_k$ be integers such that $a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 3$. For any j with $1 \leq j \leq k$, we have $\frac{n_k}{\alpha(G_{j,k})} < a_j$.*

Proof. Let $1 \leq m \leq k$ be an integer. By induction on m, we will prove that $a_j \alpha(G_{j,m}) - n_m \geq 1$ for all $1 \leq j \leq m$. Letting $m = k$, this will establish the lemma.

We proceed by induction on m. The base case $m = 1$ is trivial, as $n_1 = a_1 - 1$, and $\alpha(G_{1,1}) = 1$. So let $m \geq 2$, and suppose the lemma is true for all indices less than m. Then, by the induction hypothesis, $a_j \alpha(G_{j,m-1}) - n_{m-1} \geq 1$ for each $1 \leq j \leq m-1$. We have

$$a_j \alpha(G_{j,m-1}) - n_{m-1} \geq 1$$

$$a_j a_m \alpha(G_{j,m-1}) - a_m n_{m-1} \geq a_m$$

$$a_j(\alpha(G_{j,m}) + 1) - n_m - 1 \geq a_m, \quad \text{by Theorem 3.1.}$$

$$a_j \alpha(G_{j,m}) - n_m \geq a_m - a_j + 1$$

$$a_j \alpha(G_{j,m}) - n_m \geq 1, \quad \text{since } a_m \geq a_j.$$

This proves the lemma for each $1 \leq j \leq m-1$. Finally for $j = m$, we have $\alpha(G_{m,m}) = n_{m-1}$, by Theorem 3.1. And so $a_m \alpha(G_{m,m}) - n_m = a_m n_{m-1} - n_m = 1$, and our induction is complete.

Thus, letting $m=k$, we have shown that $a_j\alpha(G_{j,k})-n_k\geq 1$, from which the desired conclusion follows. ■

Proposition 4.6. *Let* $a_1,a_2,\ldots,a_k$ *be integers such that* $a_k\geq a_{k-1}\geq\cdots\geq a_1\geq 3$. *Then* $r_f(a_1,a_2,\ldots,a_k)>n_k$.

Proof. To establish this result, it suffices to find a k-edge coloring of K_{n_k} such that if H_j is the subgraph induced by color j, then $\omega_f(H_j)<a_j$, for all j.

Let $H_j=G_j(a_1,a_2,\ldots,a_k)=G_{j,k}$ for each $1\leq j\leq k$. By Lemma 2.1, the H_j's induce a k-edge coloring of K_{n_k}. Since $G_{j,k}$ is star extremal, Theorem 3.2 and Lemma 2.10 imply that $\omega_f(G_{j,k})=\chi_f(G_{j,k})=\frac{n_k}{\alpha(G_{j,k})}$. And then applying Lemma 4.5, we have $\omega_f(G_{j,k})<a_j$, for each $1\leq j\leq k$. This completes the proof. ■

We found an explicit formula for $\alpha(G)$, for an infinite family of circulant graphs $G=G_{j,k}$. We also proved that every graph in this family is star extremal. Using these theorems, we found a precise formula for the generalized fractional Ramsey number $r_f(a_1,a_2,\ldots,a_k)$, for any k-tuple of positive integers $(a_1,a_2,\ldots,a_k)$.

We remark that $r_f(a_1,a_2,\ldots,a_k)$ is defined for any k-tuple of positive *real* numbers. While we found an explicit formula for this function when each a_i is an integer, it would be interesting to see if a formula can also be developed, when each a_i is a real number. A formula is known [12, 16] for the $k=2$ case, but not for any $k\geq 3$. We leave this as an open problem.

5. AN APPLICATION TO INTEGER DISTANCE GRAPHS

To conclude the article, we give another application of the results in Section 3, by determining the chromatic number of an infinite family of integer distance graphs.

If S is a subset of the positive integers, then the *integer distance graph* $G(\mathbb{Z},S)$ is defined to be the graph with vertex set $\mathbb{Z}$, where two vertices u and v are adjacent iff $|u-v|\in S$. Thus, S is the set of forbidden distances with respect to coloring the integers on the real line. In a way, we can regard the integer distance graph $G(\mathbb{Z},S)$ as the infinite analogue of the circulant $C_{n,S}$.

The distance graph, first introduced by Eggleton et al. [7], is motivated by the well-known Hadwiger–Nelson problem which asks for the minimum number of colors needed to color all points of the plane such that points at unit distances receive different colors. This problem is equivalent to determining the chromatic number of $G(\mathbb{R}^2,\{1\})$, which is known to be at least 4 and at most 7. A comprehensive survey of this well-studied problem appears in [3].

Motivated by the plane coloring problem, we can consider the analogue to the one-dimensional case by investigating the chromatic numbers of distance graphs on the real line $\mathbb{R}$ and the integer set $\mathbb{Z}$. A particularly interesting problem is determining the value of $\chi(G(\mathbb{Z},S))$ for a given set S. Much work has been done on this problem [1, 2, 7, 11, 14, 18, 19, 23].

Most of the known formulas for $\chi(\mathbb{Z},S)$ occur when S is a small set or when S is a highly structured set, such as an arithmetic sequence. We generalize many of these results in this concluding section, by determining a formula for $\chi(\mathbb{Z},S)$ when $S=S_{j,k}$. This gives us explicit values of $\chi(\mathbb{Z},S)$ for a new (infinite) family of sets S, and extends much of what is currently known. Specifically, we prove the following theorem which gives us our desired formula for $\chi(\mathbb{Z},S)$, for each $S=S_{j,k}$.

Theorem 5.1. *Let $a_1,a_2,\dots,a_k$ be integers such that $a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 3$. Then, $\chi(\mathbb{Z},S_{j,k})=a_j$ whenever $1\leq j\leq k-1$ and $k\geq 3$.*

Before we prove Theorem 5.1, we require a trivial lemma.

Lemma 5.2. *Let $C_{n,S}$ be a circulant, where $S\subseteq\{1,2,\dots,\lfloor\frac{n}{2}\rfloor\}$. Then, $\chi(\mathbb{Z},S)\leq\chi(C_{n,S})$.*

Proof. The map $x\to x(\bmod n)$ is a graph homomorphism from $G(\mathbb{Z},S)$ to $C_{n,S}$. Thus, if $C_{n,S}$ is k-colorable then so is $G(\mathbb{Z},S)$. ■

There are infinitely many sets S for which equality does not hold. As a simple example, consider the case $n=4$ and $S=\{1,2\}$. Then $\chi(\mathbb{Z},S)=3$, while $\chi(C_{n,S})=\chi(K_4)=4$.

Now we are ready to prove Theorem 5.1.

Proof. By Theorem 3.2, $G_{j,k}$ is star extremal. By Lemma 2.10 and 4.5, $\chi_c(G_{j,k})=\frac{n_k}{\alpha(G_{j,k})}<a_j$. Therefore, $\chi(G_{j,k})=\lceil\chi_c(G_{j,k})\rceil\leq a_j$, by Theorem 2.7. Finally, Lemma 5.2 implies that $\chi(\mathbb{Z},S_{j,k})\leq\chi(C_{n_k,S_{j,k}})=\chi(G_{j,k})\leq a_j$.

To complete the proof, we need to prove that there is no (a_j-1) coloring of $\chi(\mathbb{Z},S_{j,k})$. We split our analysis into two subcases: when $j=1$, and when $2\leq j\leq k-1$.

Subcase 1. $j=1$.

On the contrary, suppose there is an (a_1-1)-coloring of $G(\mathbb{Z},S_{1,k})$. We note that $S_{1,2}=\{1,2,\dots,a_1-2\}$. Note that an (a_1-1)-coloring of $G(\mathbb{Z},S_{1,2})$ must be unique, up to a permutation of colors. In fact, the coloring can be characterized very easily as follows: u and v have the same color iff $u\equiv v(\text{mod } a_1-1)$.

By Proposition 2.3(b), $S_{1,2}\subseteq S_{1,k}$, and so any (a_1-1)-coloring of $G(\mathbb{Z},S_{1,k})$ must have the property that u and v have the same color iff $u\equiv v(\text{mod } a_1-1)$. So in any proper coloring, the vertices 0 and $(a_1-1)(a_2-1)$ must be colored the same, and hence $(a_1-1)(a_2-1)\notin S_{1,k}$, for all $k\geq 2$.

We claim that $n_2-n_1+1\in S_{1,3}$. To see this, note that $a_1-2=n_1-1\in S_{1,2}$. Since $S_{1,3}=\pm S_{1,2}(\bmod n_2)$, we have $n_2-(n_1-1)\in S_{1,3}$. Thus, $n_2-n_1+1\in S_{1,k}$ for all $k\geq 3$.

However, $n_2-n_1+1=(a_2n_1-1)-n_1+1=n_1(a_2-1)=(a_1-1)(a_2-1)\notin S_{1,k}$. And this is a contradiction for all $k\geq 3$. Therefore, no such (a_1-1)-coloring exists.

Subcase 2. $2\leq j\leq k-1$.

On the contrary, suppose there is an (a_j-1)-coloring of $G(\mathbb{Z},S_{j,k})$. Let H_j be the restriction of this graph to the vertices $\{0,1,2,\dots,(a_j-1)n_{j-1}\}$. By the Pigeonhole Principle, there must be $n_{j-1}+1$ vertices in H_j that appear in the same color class. Let these vertices be $v_1, v_2, \dots, v_m$, arranged in increasing order, where $m = n_{j-1}+1$. Thus, $v_m - v_1 \leq (a_j-1)n_{j-1}$.

Let $u_i = v_{i+1} - v_i$ for each $1 \leq i \leq m-1$. Since v_i and v_{i+1} belong to the same color class, $u_i \notin S_{j,k}$. By Proposition 2.3(b), $S_{j,j+1} \subseteq S_{j,k}$, and so $u_i \notin S_{j,j+1}$. By Proposition 2.3(e), $S_{j,j+1} = \{n_{j-1}, n_{j-1}+1, \dots, n_j - n_{j-1}\}$, so either $1 \leq u_i < n_{j-1}$ or $u_i > n_j - n_{j-1}$. First, suppose some $u_p \geq n_j - n_{j-1}+1$, for some $1 \leq p \leq m-1$. Since each of the other u_i's are at least 1, we have

$$\begin{aligned} v_m - v_1 &= \sum_{i=1}^{m-1} u_i \\ &\geq (n_j - n_{j-1}+1)+(m-2) \\ &= (n_j - n_{j-1}+1)+(n_{j-1}+1-2) \\ &= n_j = a_j n_{j-1} - 1 \\ &> (a_j-1)n_{j-1}. \end{aligned}$$

Since $v_m - v_1 \leq (a_j-1)n_{j-1}$, we have our desired contradiction. This shows that $u_i < n_{j-1}$, for each $1 \leq i \leq m-1$.

For each $1 \leq i \leq m$, define $w_i = v_i - v_1$. Since v_1 and v_i belong to the same color class, $w_i = v_i - v_1 \notin S_{j,j+1}$. By definition, $w_i \notin S_{j,j+1}$ for each i. Consider the strictly increasing sequence $\{w_1, w_2, \dots, w_m\}$. Since $w_m = v_m - v_1 \geq m-1 = n_{j-1}$ and $w_m \notin S_{j,j+1}$, we must have $w_m > n_j - n_{j-1}$.

Since $w_1 = 0 < n_{j-1}$ and $w_m > n_j - n_{j-1}$, there must exist a unique index r for which $w_r < n_{j-1}$ and $w_{r+1} > n_j - n_{j-1}$. For this r, we have $u_r = v_{r+1} - v_r = w_{r+1} - w_r \geq (n_j - n_{j-1}+1) - (n_{j-1}-1) = n_j - 2n_{j-1}+2 = (a_j-2)n_{j-1}+1 > n_{j-1}$. And this contradicts our claim that every $u_i < n_{j-1}$.

In both subcases, we have proven that no (a_j-1)-coloring exists in $G(\mathbb{Z}, S_{j,k})$. Therefore, we have shown that $\chi(\mathbb{Z}, S_{j,k}) = a_j$ for all $k \geq 3$ and $1 \leq j \leq k-1$. This completes the proof of Theorem 5.1. ■

Therefore, we have successfully verified our formula for $\chi(\mathbb{Z}, S_{j,k})$, for every possible generating set $S_{j,k}$ in our construction. As a corollary of Theorem 5.1, we can quickly derive the chromatic number for every graph in this infinite family $G_{j,k} = C_{n_k, S_{j,k}}$.

Corollary 5.3. *Let $a_1, a_2, \dots, a_k$ be integers such that $a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 3$. Then,*

$$\chi(G_{j,k}) = \begin{cases} a_j - 1 & \text{if } j = k = 1 \\ a_j & \text{otherwise} \end{cases}$$

Proof. By Theorem 3.2, $G_{j,k}$ is star extremal. Therefore, $\chi_c(G_{j,k}) = \chi_f(G_{j,k}) = \frac{n_k}{\alpha(G_{j,k})} < a_j$ by Lemma 2.10 and 4.5. By Theorem 2.7, $\chi(G_{j,k}) = \lceil \chi_c(G_{j,k}) \rceil \leq a_j$. This identity holds for all $1 \leq j \leq k$.

In Theorem 5.1, we showed that $\chi(\mathbb{Z},S_{j,k})=a_j$ whenever $1\leq j\leq k-1$ and $k\geq 3$. In this case, we must have $\chi(G_{j,k})=a_j$, since $a_j\geq\chi(G_{j,k})=\chi(C_{n_k,S_{j,k}})\geq\chi(\mathbb{Z},S_{j,k})=a_j$ by Lemma 5.2.

Now consider the other cases. If $j=k\geq 1$, then $\chi_c(G_{j,k})=\chi_f(G_{j,k})=\frac{n_k}{\alpha(G_{j,k})}=\frac{n_j}{\alpha(G_{j,j})}=\frac{a_jn_{j-1}-1}{n_{j-1}}=a_j-\frac{1}{n_{j-1}}$, by Theorem 3.1. If $j=k=1$, then $n_{j-1}=1$ and so $\chi_c(G_{j,k})=a_1-1$. Hence, $\chi(G_{j,k})=\lceil\chi_c(G_{j,k})\rceil=a_1-1$. If $j=k>1$, then $n_{j-1}>1$ and so $\chi(G_{j,k})=\lceil\chi_c(G_{j,k})\rceil=\lceil a_j-\frac{1}{n_{j-1}}\rceil=a_j$.

Finally, consider the case $(j,k)=(1,2)$. By Theorem 3.1, $\alpha(G_{1,2})=a_2\alpha(G_{1,1})-1=a_2-1$. Therefore, $\chi_c(G_{j,k})=\chi_f(G_{j,k})=\frac{n_2}{\alpha(G_{1,2})}=\frac{a_2n_1-1}{a_2-1}=\frac{a_2(a_1-1)-1}{a_2-1}$. Since $3\leq a_1\leq a_2$, we have $a_1-1<\frac{a_2(a_1-1)-1}{a_2-1}<a_1$. This implies that $\chi(G_{j,k})=\lceil\chi_c(G_{j,k})\rceil=a_1$.

Therefore, in all cases, we have shown that $\chi(G_{j,k})=a_j$, with the exceptional case $\chi(G_{j,k})=a_1-1$ when $j=k=1$. ■

We conclude this article by remarking that it would be worthwhile to determine other families of circulant graphs for which $\alpha(G)$ can be computed explicitly. This will almost certainly give rise to new families of star-extremal graphs, as well as additional formulas for the chromatic number of integer distance graphs.

ACKNOWLEDGMENTS

The authors thank the referees for their insightful comments, as well as the Natural Sciences and Engineering Council for its financial support of this research.

REFERENCES

[1] G. J. Chang, L. Huang, and X. Zhu, The circular chromatic number and fractional chromatic number of distance graphs, Eur J Combin 19 (1998), 423–431.

[2] J. J. Chen, G. J. Chang, and K. C. Huang, Integral distance graphs, J Graph Theory 25 (1997), 287–294.

[3] K. B. Chilakamarri, The unit-distance graph problem: a brief survey and some new results, Bull Inst Combin Appl 8 (1993), 39–60.

[4] V. Chvátal, Linear Programming, W. H. Freeman and Company, New York, 1983.

[5] B. Codenotti, I. Gerace, and S. Vigna, Hardness results and spectral techniques for combinatorial problems on circulant graphs, IEEE Trans Comput 48 (1999), 345–351.

[6] K. Collins, Circulants and sequences, SIAM J Discrete Math 11 (1998), 330–339.

[7] R. B. Eggleton, P. Erdös, and D. K. Skilton, Colouring the real line, J Combin Theory Ser B 39 (1985), 86–100.

[8] G. Gao and X. Zhu, Star extremal graphs and the lexicographic product, Discrete Math 152 (1996), 147–156.

[9] P. Hell and N. Nesetril, Graphs and Homomorphisms, Oxford University Press, Oxford, 2004.
[10] R. Hoshino, Independence polynomials of circulant graphs, Ph.D. Thesis, Dalhousie University, 2007.
[11] L. Huang and G. J. Chang, Circular chromatic numbers of distance graphs with distance sets missing multiples, Eur J Combin 21 (2000), 241–248.
[12] M. Jacobson, G. M. Levin, and E. R. Scheinerman, On fractional Ramsey numbers, Discrete Math 176 (1997), 159–175.
[13] R. Karp, Reducibility among combinatorial problems, In: Complexity of Computer Computations (E. Miller and J. W. Thatchers, Eds.), Plenum Press, New York, 1972, pp. 85–103.
[14] A. Kemnitz and M. Marangio, Chromatic numbers of integer distance graphs, Discrete Math 233 (2001), 239–246.
[15] S. Klavžar, On the fractional chromatic number and the lexicographic product of graphs, Discrete Math 185 (1998), 259–263.
[16] G. M. Levin, Selected topics in fractional graph theory, Ph.D. Dissertation, Johns Hopkins University, Baltimore, MD, 1997.
[17] K. W. Lih, D. D-F. Liu, and X. Zhu, Star extremal circulant graphs, SIAM J Discrete Math 12 (1999), 491–499.
[18] W. Lin, Some star extremal circulant graphs, Discrete Math 271 (2003), 169–177.
[19] D. D-F. Liu and X. Zhu, Distance graphs with missing multiples in the distance sets, J Graph Theory 30 (1999), 245–259.
[20] F. P. Ramsey, On a problem of formal logic, Proc London Math Soc 30 (1930), 264–286.
[21] E. R. Scheinerman and D. H. Ullman, Fractional Graph Theory: A Rational Approach to the Theory of Graphs, Wiley, New York, 1997.
[22] A. Vince, Star chromatic number, J Graph Theory 12 (1988), 551–559.
[23] M. Voigt and H. Walther, On the chromatic number of special distance graphs, Discrete Math 97 (1991), 395–397.
[24] X. Zhu, Star chromatic numbers and products of graphs, J Graph Theory 16 (1992), 557–569.