Independence polynomials of circulants with an application to music

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ABSTRACT

The *independence polynomial* of a graph G is the generating function $I(G, x) = \sum_{k \geq 0} i_k x^k$, where i_k is the number of independent sets of cardinality k in G. We show that the problem of evaluating the independence polynomial of a graph at any fixed non-zero number is intractable, even when restricted to circulants. We provide a formula for the independence polynomial of a certain family of circulants, and its complement. As an application, we derive a formula for the number of chords in an n-tet musical system (one where the ratio of frequencies in a semitone is $2^{1/n}$) without 'close' pitch classes.

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1. Introduction

Let G be a graph. A subset T of the vertex set of G is an *independent set* if no two vertices of T are adjacent in G. We can encode the number of independent sets of each cardinality by a generating function.

Definition 1.1 (*[17]*). The *independence polynomial* of a graph G on n vertices is $I(G, x) = \sum_{k=0}^{n} i_k x^k$, where i_k is the number of independent sets of cardinality k in G.

By definition, the independence number $\alpha(G)$ of a graph G is equal to $\deg(I(G, x))$, the degree of the independence polynomial $I(G, x)$.

For example, the independence polynomial of the 6-cycle C_6 is given by

$$I(C_6, x) = 1 + 6x + 9x^2 + 2x^3,$$

as C_6 has $i_0 = 1$ (the empty set), $i_1 = 6$, $i_2 = 9$ (the number of non-edges of G), and $i_3 = 2$. The latter follows as there are precisely two independent sets of cardinality 3, namely $\{0, 2, 4\}$ and $\{1, 3, 5\}$.

A variety of graph polynomials, such as chromatic polynomials, matching polynomials, characteristic polynomials, have been well studied. Independence polynomials have been investigated in a number of papers [5–8,14,16–22].

One highly structured (and well known) family of graphs are *circulants*. Given $n \geq 1$ and $S \subseteq \mathbb{Z}_n - \{0\}$ with $-S = \{-s : s \in S\} = S$, the *circulant* $C_{n,S}$ of order n with *generating set* S is a graph on $V = \mathbb{Z}_n$ such that for $u, w \in V$, uw is an edge of $C_{n,S}$ if and only if $u - w \in S$. Such graphs are regular and vertex transitive, and arise in a variety of graph applications. We study here the independence polynomials of circulants.

In Section 2, we show that for any $t \neq 0$, the problem of evaluating the independence polynomial $I(G, x)$ at $x = t$ is intractable, even when restricted to circulants. In Section 3, we consider circulants of the form $C_n^d = C_{n, \{\pm 1, \pm 2, \ldots, \pm d\}}$, where adjacent vertices are separated by a circular distance of at most d. For each pair (n, d), we compute an explicit formula for the independence polynomial $I(C_n^d, x)$ as well as $I(\overline{C_n^d}, x)$, the independence polynomial of its complement. In Section 4, we use

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the theorems from the previous section to provide an application of independence polynomials to music. Specifically, we establish a formula for the number of chords in an n-tet musical system (whose semitone corresponds to a ratio of $2^{1/n}$ without 'close' pitch classes).

In what follows, for any polynomial $P(x)$, we shall denote by $[\boldsymbol{x^k}]\boldsymbol{P(x)}$ the coefficient of the x^k term in $P(x)$. In general we follow [12] for graph-theoretic terminology. For discussion of relevant computational complexity, we refer the reader to [15,23].

2. The intractability of evaluating the independence polynomial at non-zero numbers

Given the independence polynomial $I(G, x)$ of a graph G, we may be interested in evaluating the polynomial at a particular point $x = t$. As an example, evaluating $I(G, x)$ at $x = 1$ gives us the total number of independent sets in the graph. Evaluating a graph polynomial at particular points has been a subject of much interest, especially for chromatic polynomials [13,23].

In general, it is *NP*-hard to determine the independence polynomial $I(G, x)$, since we know that evaluating $\alpha(G)$ is *NP*-hard [15]. Thus, it is not computationally efficient to solve the problem by first computing the independence polynomial.

We wish to determine the complexity of evaluating $I(G, t)$ for an arbitrary number $t \in \mathbb{C}$. For $t = 0$ it is obviously polynomial (it is 1), so we only consider $t \neq 0$. The equivalent problem for chromatic polynomials has already been solved in [23], where it was shown that evaluating the chromatic polynomial is #*P*-hard for all $t \notin \{0, 1, 2\}$ (for these values, the evaluation can be easily be computed in polynomial time).

We now give a complete solution to the evaluation problem for independence polynomials, even when restricted to circulants: we prove that for any $t \in \mathbb{C} - \{0\}$, it is #*P*-hard to evaluate $I(G, t)$. Furthermore, if $t = 1$, then the problem is #*P*-complete. First, we require a definition and a theorem on the lexicographic product of two graphs.

Definition 2.1. For any two graphs G and H, the *lexicographic product* is a new graph $G[H]$ with vertex set $V(G) \times V(H)$ such that any two vertices (g, h) and (g', h') in $G[H]$ are adjacent iff $(g \sim g')$ or $(g = g'$ and $h \sim h')$.

The following theorem shows that $I(G[H], x)$ can be calculated directly from $I(G, x)$ and $I(H, x)$.

Theorem 2.2 (*[6]*). *For any graphs G and H,*

$$I(G[H], x) = I(G, I(H, x) - 1).$$

We can prove our theorem on the computational complexity of evaluating $I(G, x)$ at $x = t$.

Theorem 2.3. *Computing $I(G, t)$ for a given number $t \in \mathbb{C} - \{0\}$ is #P-hard, even when restricted to circulant graphs G.*

Proof. Suppose there exists a number $t \neq 0$ for which every $I(G, t)$ can be evaluated in polynomial-time, whenever G is a circulant. In other words, given any circulant graph G on n vertices, there exists an $O(n^k)$ algorithm to compute $I(G, t)$ for some constant k.

Let G be a fixed circulant on n vertices. For each $1 \leq m \leq n + 1$, define H_m to be the lexicographic product graph $G[K_m]$. By a theorem in [4], the lexicographic product of two circulants is always a circulant. Since K_m is (trivially) a circulant, $G[K_m]$ is as well. By our assumption, there is an $O(m^k n^k) \leq O(n^{2k})$ algorithm to compute the value of $I(H_m, t)$.

The construction of each H_m creates $nm \leq n^2 + n$ vertices and decides if each pair of vertices is adjacent in H_m. The number of pairs of vertices in H_m is at most $\binom{n^2+n}{2} = O(n^4)$, and so constructing each H_m can be done in polynomial-time.

By Theorem 2.2, $I(H_m, x) = I(G, I(K_m, x) - 1) = I(G, mx)$. Therefore, $I(H_m, t) = I(G, mt)$ for all $1 \leq m \leq n + 1$. We know that there is an $O(m^k n^k) \leq O(n^{2k})$ algorithm to compute the value of $I(H_m, t)$, for each m. Therefore, it takes $O(n^{2k+1})$ steps to evaluate $I(G, x)$ for each $x = mt$. Since $t \neq 0$, these n values of x are distinct.

We know that the independence polynomial of G is $I(G, x) = i_0 + i_1 x + i_2 x^2 + \cdots + i_n x^n$, for some integers i_k. (Note that $\deg(I(G, x)) \leq |G| = n$). Letting $x = mt$ for each $1 \leq m \leq n + 1$, we have a system of $n + 1$ equations and $n + 1$ unknowns.

$$i_0 + i_1 t + i_2 t^2 + \cdots + i_n t^n = I(G, t)$$
$$i_0 + i_1(2t) + i_2(2t)^2 + \cdots + i_n(2t)^n = I(G, 2t)$$
$$\vdots$$
$$i_0 + i_1(n + 1)t + i_2((n + 1)t)^2 + \cdots + i_n((n + 1)t)^n = I(G, (n + 1)t).$$

This system has a unique solution iff the matrix

$$M = \begin{pmatrix} 1 & t & t^2 & \cdots & t^n \\ 1 & 2t & (2t)^2 & \cdots & (2t)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (n+1)t & ((n+1)t)^2 & \cdots & ((n+1)t)^n \end{pmatrix}$$
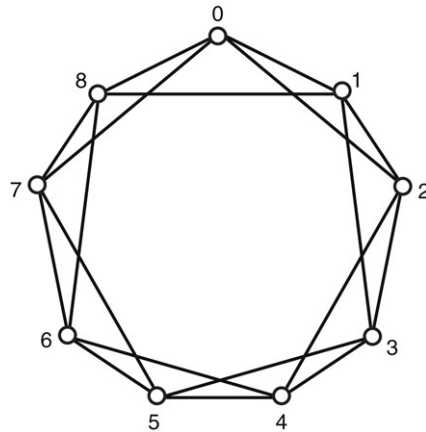
Fig. 1. The cycle power graph C_9^2.

has a non-zero determinant. M is a *Vandermonde* matrix, and the formula for its determinant is

$$\det(M) = t^{\binom{n+1}{2}} \prod_{k=1}^{n} k!.$$

Since $\det(M) \neq 0$, this system has a unique solution $(i_0, i_1, \ldots, i_n)$. This system of equations can be solved in $O(n^3)$ time using Gaussian elimination, and so each of the i_k's can be determined in polynomial time, which in turn, gives us the independence polynomial $I(G, x)$.

For any graph G, we have found an $O(n^4) + O(n^{2k+1}) + O(n^3)$ algorithm to determine the formula for $I(G, x)$. Since $\deg(I(G, x)) = \alpha(G)$, we have shown that $\alpha(G)$ can be computed in polynomial-time for any circulant G. This contradicts the result [11] that no such algorithm exists. Since it is *NP*-hard to evaluate $\alpha(G)$ for an arbitrary circulant G [11], we conclude that it is #*P*-hard to evaluate $I(G, t)$, for all non-zero $t \in \mathbb{C}$. □

We note that when $t = 1$, it is #*P*-complete to evaluate $I(G, t)$. This follows since $I(G, 1)$ simply represents the total number of independent sets in G, and independent sets are recognizable in polynomial time. Thus, for the specific case $t = 1$, the problem of evaluating $I(G, t)$ is #*P*-complete.

In light of Theorem 2.3, it is of interest to find families of graphs (and indeed families of circulants) for which we can find explicit formulas for their independence polynomials.

3. The independence polynomials of a family of circulants

Let $d \geq 1$ be a fixed integer. In the circulant C_n^d, two vertices are adjacent iff their circular distance is at most d, that is uw is an edge iff $|u - w| \leq d$. The graph C_n^d is also known as the dth power of the cycle C_n. Powers of cycles have been a rich study of investigation [2,3,24–26], with important connections to the analysis of perfect graphs [1,9,10,27]. The graph C_9^2 is illustrated in Fig. 1.

For notational convenience, we adopt the following definition, since $d \geq 1$ will always be fixed.

Definition 3.1. Let $d \geq 1$ be a fixed integer. For each n, set $A_n = C_n^d$.

In this section, we determine a formula for $I(A_n, x)$, for all $n \geq 1$. By definition, note that A_n is the complete graph K_n for $n \leq 2d + 1$, i.e., $I(A_n, x) = I(K_n, x) = 1 + nx$. Thus, we may assume that $n \geq 2d + 2$.

Lemma 3.2. $I(A_n, x) = I(A_{n-1}, x) + x \cdot I(A_{n-d-1}, x)$, for all $n \geq 2d + 2$.

Proof. Since $n \geq 2d + 2$, we have $\alpha(A_n) \geq 2$. We see trivially that the x^0 and x^1 coefficients are equal in the given identity. So fix $k \geq 2$. We will show that the x^k coefficients are equal as well.

Let $\{v_1, v_2, \ldots, v_k\}$ be an independent set of cardinality $k \geq 2$ in A_n, with $0 \leq v_1 < v_2 < \cdots < v_k \leq n - 1$. Since the circular distance exceeds d for all non-adjacent vertices u and v in A_n, we have $v_{i+1} - v_i > d$ for all $1 \leq i \leq k - 1$, and $n + (v_1 - v_k) > d$. This can be seen by placing n points equally around a circle, and noticing that each (adjacent) pair of chosen vertices is separated by distance greater than d.

We classify our independent sets $\{v_1, v_2, \ldots, v_k\}$ of A_n into two families:

(a) $S_1 = \{\{v_1, v_2, \ldots, v_k\} \text{ independent in } A_n : v_k - v_{k-1} = d + 1\}$.
(b) $S_2 = \{\{v_1, v_2, \ldots, v_k\} \text{ independent in } A_n : v_k - v_{k-1} > d + 1\}$.

Since $S_1 \cap S_2 = \emptyset$, it follows that $[x^k]A_n = |S_1| + |S_2|$. We will show that $|S_1| = [x^{k-1}]A_{n-d-1}$ and $|S_2| = [x^k]A_{n-1}$.

Case 1: Proving $|S_1| = [x^{k-1}]A_{n-d-1}$.

We establish a bijection ϕ between S_1 and the set of $(k-1)$-tuples that are independent in A_{n-d-1}. This will prove that $|S_1| = [x^{k-1}]A_{n-d-1}$. For each element of S_1, define

$$\phi(v_1, v_2, \ldots, v_k) = \{v_1, v_2, \ldots, v_{k-1}\}.$$

Since $v_k = v_{k-1} + (d+1)$, ϕ is one-to-one. Construct the graph A'_n by contracting all of the vertices from the set $\{v_{k-1}+1, v_{k-1}+2, \ldots, v_k\}$ to v_{k-1}. Then $A'_n \simeq A_{n-d-1}$. We claim that $\phi(v_1, v_2, \ldots, v_k)$ is an independent set of A'_n iff $\{v_1, v_2, \ldots, v_k\}$ is an element of S_1.

Note that $\phi(v_1, v_2, \ldots, v_k)$ is an independent set of A'_n iff

(a) $v_{i+1} - v_i > d$ for $1 \leq i \leq k-2$.
(b) $(n-d-1) + v_1 - v_{k-1} > d$.

Also, $\{v_1, v_2, \ldots, v_k\}$ is an element of S_1 iff

(a) $v_{i+1} - v_i > d$ for $1 \leq i \leq k-2$.
(b) $v_k - v_{k-1} = d+1$.
(c) $n + v_1 - v_k > d$.

We now show that these two sets of conditions are equivalent.

Note that the condition $v_{i+1} - v_i > d$ for $1 \leq i \leq k-2$ is true in both cases. If $\phi(v_1, v_2, \ldots, v_k) = \{v_1, v_2, \ldots, v_{k-1}\}$ is an independent set of A'_n, then $(n-d-1) + v_1 - v_{k-1} > d$. Let $v_k = v_{k-1} + (d+1)$. Then, $\{v_1, v_2, \ldots, v_{k-1}, v_k\}$ is an independent set of A_n, since $(n-d-1) + v_1 - (v_k - (d+1)) > d$, or $n + v_1 - v_k > d$. Therefore, $\{v_1, v_2, \ldots, v_k\}$ is an element of S_1.

Now we prove the converse. If $\{v_1, v_2, \ldots, v_k\}$ is an element of S_1, then $v_k - v_{k-1} = d+1$ and $n + v_1 - v_k > d$. Adding, this implies that $(v_k - v_{k-1}) + (n + v_1 - v_k) > 2d + 1$, or $(n-d-1) + v_1 - v_{k-1} > d$. Hence, $\phi(v_1, v_2, \ldots, v_k)$ is an independent set of A'_n.

Therefore, we have established that ϕ is a bijection between the sets in S_1 and the independent sets of cardinality $k-1$ in $A'_n \simeq A_{n-d-1}$. We conclude that $|S_1| = [x^{k-1}]A_{n-d-1}$.

Case 2: Proving $|S_2| = [x^k]A_{n-1}$.

We now establish a bijection φ between S_2 and the set of independent k-tuples in A_{n-1}. For each element $(v_1, v_2, \ldots, v_{k-1}, v_k)$ of S_2, define

$$\varphi(v_1, v_2, \ldots, v_{k-1}, v_k) = \{v_1, v_2, \ldots, v_{k-1}, v_k - 1\}.$$

Observe that φ is one-to-one. Construct the graph A''_n by contracting v_k to $v_k - 1$. Then, $A''_n \simeq A_{n-1}$. We claim that $\varphi(v_1, v_2, \ldots, v_k)$ is an independent set of A''_n iff $\{v_1, v_2, \ldots, v_k\}$ is an element of S_2.

Note that $\varphi(v_1, v_2, \ldots, v_k)$ is an independent set of A''_n iff

(a) $v_{i+1} - v_i > d$ for $1 \leq i \leq k-2$.
(b) $(v_k - 1) - v_{k-1} > d$.
(c) $(n-1) + v_1 - (v_k - 1) > d$.

Also, $\{v_1, v_2, \ldots, v_k\}$ is an element of S_2 iff

(a) $v_{i+1} - v_i > d$ for $1 \leq i \leq k-2$.
(b) $v_k - v_{k-1} > d+1$.
(c) $n + v_1 - v_k > d$.

Clearly, these sets of conditions are equivalent. Therefore, we have established that φ is a bijection between the sets in S_2 and the independent sets of cardinality k in $A''_n \simeq A_{n-1}$. We conclude that $|S_2| = [x^k]A_{n-1}$.

Therefore, we have shown that $[x^k]A_n = [x^{k-1}]A_{n-d-1} + [x^k]A_{n-1}$, which implies that $I(A_n, x) = I(A_{n-1}, x) + x \cdot I(A_{n-d-1}, x)$. $\quad\square$

Now we find an explicit formula for $I(A_n, x)$.

Theorem 3.3. *Let $n \geq d+1$. Then,*

$$I(A_n, x) = I(C_n^d, x) = \sum_{k=0}^{\lfloor \frac{n}{d+1} \rfloor} \frac{n}{n - dk} \binom{n - dk}{k} x^k.$$

Proof. By Lemma 3.2, $I(A_n, x) = I(A_{n-1}, x) + x \cdot I(A_{n-d-1}, x)$, for $n \geq 2d + 2$. We will prove the theorem using generating functions.

Let $f_n = \begin{cases} I(A_n, x) & \text{for } n \geq d + 1 \\ 1 & \text{for } 1 \leq n \leq d \\ d + 1 & \text{for } n = 0. \end{cases}$

Each f_n is a polynomial in x. First, we verify that $f_n = f_{n-1} + x f_{n-d-1}$, for all $n \geq d+1$. This recurrence is true for $n \geq 2d+2$, by Lemma 3.2. For $d + 2 \leq n \leq 2d + 1$, we have $f_n = 1 + nx = (1 + (n-1)x) + x \cdot 1 = f_{n-1} + x f_{n-d-1}$. Finally, for $n = d + 1$, we have $f_{d+1} = 1 + (d + 1)x = f_d + x f_0$. Thus, $f_n = f_{n-1} + x f_{n-d-1}$, for all $n \geq d + 1$.

Let $F(x, y) = \sum_{p=0}^{\infty} f_p y^p$. For each $n \geq d + 1$, we will show that

$$[x^k y^n]F(x, y) = \binom{n - dk}{k} + d \binom{n - dk - 1}{k - 1} = \frac{n}{n - dk} \binom{n - dk}{k}.$$

Since $f_n = f_{n-1} + x f_{n-d-1}$, for all $n \geq d + 1$, we have

$$\sum_{n=d+1}^{\infty} f_n y^n = \sum_{n=d+1}^{\infty} f_{n-1} y^n + \sum_{n=d+1}^{\infty} f_{n-d-1} x y^n$$

$$F(x, y) - \sum_{n=0}^{d} f_n y^n = y \left(F(x, y) - \sum_{n=0}^{d-1} f_n y^n \right) + x y^{d+1} F(x, y)$$

$$F(x, y)(1 - y - xy^{d+1}) = f_0 + f_1 y + \sum_{n=2}^{d} f_n y^n - f_0 y - \sum_{n=1}^{d-1} f_n y^{n+1}$$

$$F(x, y)(1 - y - xy^{d+1}) = f_0 + f_1 y + \sum_{n=2}^{d} y^n - f_0 y - \sum_{n=2}^{d} y^n$$

$$F(x, y)(1 - y - xy^{d+1}) = (d + 1) + y - (d + 1)y$$

$$F(x, y) = (d + 1 - dy)(1 - y - xy^{d+1})^{-1}$$

$$F(x, y) = (d + 1 - dy) \sum_{t=0}^{\infty} (y + xy^{d+1})^t$$

$$F(x, y) = (d + 1 - dy) \sum_{t=0}^{\infty} y^t (1 + xy^d)^t$$

$$F(x, y) = (d + 1 - dy) \sum_{t=0}^{\infty} \sum_{u=0}^{\infty} \binom{t}{u} x^u y^{t+du}$$

$$F(x, y) = (d + 1) \sum_{t,u=0}^{\infty} \binom{t}{u} x^u y^{t+du} - d \sum_{t,u=0}^{\infty} \binom{t}{u} x^u y^{t+du+1}.$$

Now we extract the $x^k y^n$ coefficient of $F(x, y)$. The last line will follow from Pascal's Identity.

$$[x^k y^n]F(x, y) = [x^k y^n](d + 1) \sum_{t,u=0}^{\infty} \binom{t}{u} x^u y^{t+du} - [x^k y^n]d \sum_{t,u=0}^{\infty} \binom{t}{u} x^u y^{t+du+1}$$

$$= (d + 1) \binom{n - dk}{k} - d \binom{n - dk - 1}{k}$$

$$= \binom{n - dk}{k} + d \left[\binom{n - dk}{k} - \binom{n - dk - 1}{k} \right]$$

$$= \binom{n - dk}{k} + d \binom{n - dk - 1}{k - 1}$$

$$= \frac{n}{n - dk} \binom{n - dk}{k}.$$

Therefore, we have proven that

$$[x^k]I(A_n, x) = [x^k y^n]F(x, y) = \frac{n}{n - dk} \binom{n - dk}{k}.$$

We note that this coefficient is non-zero precisely when $n - dk \geq k$, which is equivalent to the condition $k \leq \frac{n}{d+1}$. Hence, $\deg(I(A_n, x)) = \lfloor \frac{n}{d+1} \rfloor$.

We conclude that $I(C_n^d, x) = I(A_n, x) = \sum_{k=0}^{\lfloor \frac{n}{d+1} \rfloor} \frac{n}{n-dk} \binom{n-dk}{k} x^k$. $\square$

We remark that our formula for the special case $d = 1$, namely

$$I(C_n, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^k,$$

has previously appeared in the literature [17], via an alternate method of proof.

Now we compute a formula for $I(\overline{C_n^d}, x)$, the independence polynomial of the complement of C_n^d. Since d will remain fixed, we introduce the following definition for notational convenience.

Definition 3.4. Let $d \geq 0$ be a fixed integer. For each $n \geq 2d+2$, define the graph B_n to be the complement of A_n. Specifically,

$$B_n = \overline{A_n} = \overline{C_n^d}.$$

Note that if $n = 2d + 2$, then B_n is the disjoint union of $d + 1$ copies of K_2, and so $I(B_n, x) = [I(K_2, x)]^{d+1} = (1 + 2x)^{d+1}$. If $n = 2d + 3$, then $B_n = B_{2d+3}$ is simply the cycle C_{2d+3}, and a formula for this independence polynomial was established in Theorem 3.3. Finally, if $n \geq 3d + 1$, then the formula for the independence polynomial is straightforward to prove. The correct formula was first established in a paper by Michael and Traves.

Proposition 3.5 ([28]). *Let $n \geq 3d + 1$. Then, $I(B_n, x) = 1 + nx(1 + x)^d$.*

Therefore, we are left with the case $2d + 4 \leq n \leq 3d$. As we will see, determining a formula for $I(B_n, x) = I(\overline{A_n}, x)$ in this case will be extremely complicated, and the proof will require many technical lemmas. First, we introduce the following definition.

Definition 3.6. For each k-tuple $\{v_1, v_2, \ldots, v_k\}$ of the vertices of a graph G on n vertices, with $0 \leq v_1 < v_2 < \cdots < v_k \leq n - 1$, the *difference sequence* is

$$(d_1, d_2, \ldots, d_k) = (v_2 - v_1, v_3 - v_2, \ldots, v_k - v_{k-1}, n + v_1 - v_k).$$

Difference sequences will be of tremendous help in counting the number of independent sets. We will carefully study the structure of these difference sequences, and determine a direct correlation to independent sets. As we did in the proof of Lemma 3.2, we can visualize difference sequences as follows: spread n vertices around a circle, and highlight the k chosen vertices $v_1, v_2, \ldots, v_k$. Now, let d_i be the distance between v_i and v_{i+1}, for each $1 \leq i \leq k$ (note: $v_{k+1} := v_1$).

In other words, the d_i's just represent the distances between each pair of highlighted vertices. By this reasoning, it is clear that $\sum_{i=1}^k d_i = n$ and that $v_j = v_1 + \sum_{i=1}^{j-1} d_i$ for each $1 \leq j \leq k$.

Instead of directly enumerating the independent sets I of B_n, it will be easier to determine all possible difference sequences D that correspond to an independent set of B_n, and then enumerate the number of independent sets corresponding to these difference sequences. For notational convenience, we introduce the following definition.

Definition 3.7. A difference sequence $D = (d_1, d_2, \ldots, d_k)$ of the circulant B_n is *valid* if no cyclic subsequence of consecutive d_i's sum to an element in $\{d + 1, d + 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

By a *cyclic* subsequence of consecutive terms, we refer to subsequences such as $\{d_{k-2}, d_{k-1}, d_k, d_1, d_2, d_3, d_4\}$. From now on, when we refer to subsequences of D, this will automatically include all cyclic subsequences.

We note that each independent set I of B_n maps to a valid difference sequence D. The following lemma is immediate from the definitions, and so we omit the proof.

Lemma 3.8. *Let the set $I = \{v_1, v_2, \ldots, v_k\}$ have the difference sequence $D = (d_1, d_2, \ldots, d_k)$. Then, I is independent in B_n iff D is valid.*

We will now describe an explicit construction of all valid difference sequences with k elements, and this will yield the total number of independent sets with cardinality k. The desired formula for $I(B_n, x)$ is the following.

Theorem 3.9. *Let (n, d) be an ordered pair with $n \geq 2d + 2$. Let $B_n = \overline{A_n} = \overline{C_n^d}$, and set $r = n - 2d - 2 \geq 0$. Then,*

$$I(B_n, x) = 1 + \sum_{l=0}^{\lfloor \frac{d}{r+2} \rfloor} \frac{n}{2l+1} \binom{d - lr}{2l} x^{2l+1} (1 + x)^{d - l(r+2)}.$$

It is easy to show that Proposition 3.5 follows immediately from Theorem 3.9. We omit the details. To prove Theorem 3.9, we require several technical combinatorial lemmas.

In the next lemma, we count the number of m-tuples $(Q_1, Q_2, \ldots, Q_m)$ with a fixed sum that contain a total of t non-zero elements among the Q_i's. In this case, each Q_i is a (possibly empty) sequence of positive integers.

Lemma 3.10. *Let $a_1, a_2, \ldots, a_m$ be non-negative integers with sum k. Then there are exactly $\binom{k}{t}$ m-tuples $(Q_1, Q_2, \ldots, Q_m)$ that contain a total of t non-zero elements among the Q_i's, where each Q_i is a (possibly empty) sequence of positive integers whose sum is at most a_i.*

Proof. Write down a string of k ones, and place $m - 1$ bars in between the ones to create the partition corresponding to the m-tuple $(a_1, a_2, \ldots, a_m)$. Now select any t of the k ones.

As an example, we demonstrate this for the case $(a_1, a_2, a_3) = (5, 6, 4)$, $m = 3$, $k = 15$, and $t = 6$.

$$1, \mathbf{1}, 1, \mathbf{1}, 1 | \mathbf{1}, \mathbf{1}, 1, 1, 1, \mathbf{1} | 1, \mathbf{1}, 1, 1.$$

Clearly, there are $\binom{k}{t}$ ways to select exactly t ones from this string. We map each selection to a unique m-tuple $(Q_1, Q_2, \ldots, Q_m)$ which contains a total of t non-zero elements among the Q_i's, so that the sum of the elements in each Q_i is at most a_i.

Consider the substring of a_i ones in the ith partition. If no elements are selected from this substring, set $Q_i = \emptyset$. Otherwise, let the selected elements in the ith partition be in positions $r_1, r_2, \ldots, r_p$, where $1 \leq r_1 < r_2 < \cdots < r_p \leq a_i$. Now define

$$Q_i = (r_2 - r_1, r_3 - r_2, \ldots, r_p - r_{p-1}, a_i + 1 - r_p).$$

In other words, each Q_i can be thought of as the *difference sequence* of the p chosen vertices in a circulant of order a_i.

In the above example, our selection of the t's corresponds to the sets $Q_1 = \{2, 2\}, Q_2 = \{1, 4, 1\}, Q_3 = \{3\}$, which contain a total of $t = 6$ non-zero elements.

Note that for each i, $\sum Q_i = a_i + 1 - r_1 \leq a_i$. This construction guarantees that each of the $\binom{k}{t}$ selections maps to a unique m-tuple $(Q_1, Q_2, \ldots, Q_m)$ with a total of t non-zero elements, so that $\sum Q_i \leq a_i$. Given such an m-tuple, we now justify that we can determine the unique way the t ones were selected from the string. For each substring of ones in the ith partition, we are given Q_i. From the above definition for Q_i, we can determine the values (or positions) of the r_j's by starting at r_p and calculating backwards. From r_p, we can uniquely compute r_{p-1}, r_{p-2}, and so on, until we have determined all of the r_j's. Since we can do this for each i, each selection of the m-tuple $(Q_1, Q_2, \ldots, Q_m)$ corresponds to a unique selection of t elements from a string of k ones. Hence, this construction is bijective, and our proof is complete. $\quad\square$

We now introduce l-constructible difference sequences. While the definition may appear contrived, it is precisely the insight we need to count the number of independent sets of B_n. We will show that every difference sequence is uniquely l-constructible, for exactly one integer $l \geq 0$. Then in our proof of Theorem 3.9, we will enumerate the number of l-constructible difference sequences to determine the number of independent sets of each cardinality.

Definition 3.11. Let D be a difference sequence of $B_n = \overline{A_n} = \overline{C_n^d}$, where $n \geq 2d + 2$. Then, for each integer $l \geq 0$, D is *l-constructible* if D can be expressed in the form

$$D = Q_1, p_1, Q_2, p_2, \ldots, Q_{2l+1}, p_{2l+1}$$

such that the following properties hold.

1. Each p_i is an integer satisfying $p_i \geq n - 2d$.
2. Each Q_i is a sequence of integers, possibly empty.
3. Let S be any (cyclic) subsequence of consecutive terms in D with sum $\sum S$. If S contains at most l of the p_i's, then $\sum S \leq d$. Otherwise, $\sum S \geq n - d$.

We now prove that every valid difference sequence can be expressed uniquely as an l-constructible sequence, for exactly one $l \geq 0$. We will then enumerate the number of l-constructible sequences for each l, which will give us the total number of valid difference sequences.

A difference sequence D of $B_n = \overline{A_n} = \overline{C_n^d}$ is valid iff no subsequence of consecutive terms adds up to an element in $\{d + 1, d + 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Since the complement of any consecutive subsequence of D is also a consecutive subsequence of D, there exists a consecutive subsequence with sum t iff there exists a consecutive subsequence with sum $n - t$. In other words, D is valid iff no subsequence of consecutive terms sums to an element in $[d + 1, n - d - 1]$.

By the third property in the definition of l-constructibility (see above), every l-constructible sequence is necessarily valid because every subsequence of consecutive terms has sum at most d or at least $n - d$, and hence falls outside of the forbidden range $[d + 1, n - d - 1]$. So every l-constructible sequence is a valid difference sequence. In the next two lemmas, we prove that every valid difference sequence is *uniquely* l-constructible, for exactly one $l \geq 0$. First, we construct an l that satisfies the conditions, and then we prove that no other l suffices.

To supplement the technical details of the following proof, let us describe our method by illustrating an example.

Consider the case $n = 89$ and $d = 40$. It is straightforward to show that the difference sequence $D = \{9, 1, 9, 1, 9, 20, 10, 19, 2, 9\}$ is valid, i.e., no subsequence of consecutive elements sums to any $S \in [41, 49]$. We prove that this difference sequence D is uniquely 2-constructible, up to cyclic permutation.

Lemma 3.12. *Let D be a valid difference sequence of B_n. Then there exists an integer $l \geq 0$ such that D is l-constructible. For this integer l, D is l-constructible in a unique way up to cyclic permutation, i.e., there is only one way to select the Q_i's and p_i's so that D is l-constructible.*

Proof. Let $D = R_1 t_1 R_2 t_2 \ldots R_m t_m$, where each $t_i \geq n - 2d$ and each R_i is a (possibly empty) sequence of terms, all of which are less than $n - 2d$. Thus, each D has a unique representation in this form, up to cyclic permutation. In our example, $n - 2d = 9$. Without loss, assume $t_1 = 20$. In this case, we *must* have $R_2 = \emptyset, t_2 = 10, R_3 = \emptyset, t_3 = 19, R_4 = \{2\}, t_4 = 9, R_5 = \emptyset, t_5 = 9$, $R_6 = \{1\}, t_6 = 9, R_7 = \{1\}, t_7 = 9$, and $R_1 = \emptyset$. In other words, we have

$$D = \underbrace{20}_{t_1}, \underbrace{10}_{t_2}, \underbrace{19}_{t_3}, \underbrace{2}_{R_4}, \underbrace{9}_{t_4}, \underbrace{9}_{t_5}, \underbrace{1}_{R_6}, \underbrace{9}_{t_6}, \underbrace{1}_{R_7}, \underbrace{9}_{t_7}.$$

Let $l \geq 0$ be the largest integer such that for any subsequence X of consecutive terms of D, $\sum X \leq d$ if X includes at most l of the t_i's. In our example, $l < 3$ since $X = \{20, 10, 19\}$ includes three of the t_i's, and $\sum X = 49 > d$. By inspection, it can be checked that $l = 2$.

For this $l \geq 0$, we prove that D is l-constructible, and that the assignment of Q_i's and p_i's is unique, up to cyclic permutation.

First suppose that $m \leq 2l$. Note that $R_1 + t_1 + R_2 + t_2 + \cdots + R_l + t_l \leq d$ since this series contains exactly l of the t_i's. Similarly, $R_{l+1} + t_{l+1} + \cdots + R_{2l} + t_{2l} \leq d$. If $m \leq 2l$, then $n = \sum D \leq 2d < n$, a contradiction. Thus, $m \geq 2l + 1$. If $m = 2l + 1$, then we can set $Q_i = R_i$ and $p_i = t_i$ for each i. Then each D is l-constructible, since $\sum D \leq d$ if D contains at most l of the p_i's, and $\sum D \geq n - d$ otherwise. It is clear that this is the *only* assignment that enables D to be l-constructible, up to cyclic permutation.

So suppose that $m > 2l + 1$. In this case, we will assign the p_i's and Q_i's from the set of t_i's and R_i's. All of the p_i's will be chosen from the set of t_i's, while all of the Q_i's will be determined from the R_i's, as well as any leftover t_i's not included among the p_i's.

By the definition of the index $l \geq 0$, there must be a subsequence X containing $l + 1$ of the t_i's such that its sum exceeds d. Since D is valid, no subsequence of consecutive terms can sum to any number in $[d + 1, n - d - 1]$. Therefore, $\sum X > d$ implies that $\sum X \geq n - d$.

Cyclically permute the elements of D so that this subsequence X appears at the front of D, i.e., redefine the R_i's and t_i's so that we have

$$t_1 + \sum R_2 + t_2 + \cdots + \sum R_{l+1} + t_{l+1} \geq n - d.$$

Then set $p_i = t_i$ for $1 \leq i \leq l + 1$ and $Q_i = R_i$ for $2 \leq i \leq l + 1$. In our example, we have $X = \{20, 10, 19\}, p_1 = 20$, $Q_2 = \emptyset, p_2 = 10, Q_3 = \emptyset$, and $p_3 = 19$. Note that this assignment of p_i's and Q_i's is necessary for D to be l-constructible: if any of these Q_i's contains a t_j term, then we will obtain a contradiction because the above subsequence X will have at most l of the p_i's, but its sum will exceed d.

If D is l-constructible, we require the chosen p_i's and Q_i's to satisfy

$$\sum Q_2 + p_2 + \cdots + \sum Q_{l+1} + p_{l+1} + \sum Q_{l+2} \leq d,$$

since this subsequence contains l of the p_i's. Also, we require

$$\sum Q_2 + p_2 + \cdots + \sum Q_{l+1} + p_{l+1} + \sum Q_{l+2} + p_{l+2} \geq n - d,$$

since this subsequence contains $l + 1$ of the p_i's.

Let $T = \sum Q_2 + p_2 + \cdots + \sum Q_{l+1} + p_{l+1}$. Then $\sum Q_{l+2} \leq d - T$ and $\sum Q_{l+2} + p_{l+2} \geq n - d - T$. Since each p_i and Q_i has already been assigned for $2 \leq i \leq l + 1$, T is a fixed integer. From these two inequalities, we claim that Q_{l+2} is uniquely determined. Note that for some $k \geq 0$, Q_{l+2} must be the first k elements of the sequence $X' = R_{l+2}, t_{l+2}, R_{l+3}, t_{l+3}, \ldots R_m, t_m, R_1$. Furthermore, p_{l+2} would have to be the *next* term, i.e., the $(k + 1)$th term of X'.

We claim that k must be the largest integer such that the first k terms of X' sum to at most $d - T$. This choice is unique because if k were not the largest integer, then $\sum Q_{l+2} + p_{l+2} \leq d - T$, and that contradicts the inequality $\sum Q_{l+2} + p_{l+2} \geq n - d - T$. Since k is uniquely determined, Q_{l+2} *must* represent the first k elements of X', in order for D to be l-constructible. Furthermore, p_{l+2} *must* be the next term in this subsequence. In our example, $T = 29, X' = \{2, 9, 9, 1, 9, 1, 9\}$, $Q_4 = \{2, 9\}$, and $p_4 = 9$.

Consider this sum $T + \sum Q_{l+2} + p_{l+2} > d$. By our choice of k, this sum exceeds d. Since D is valid, this sum must be at least $n - d$, since this total represents the sum of a subsequence of consecutive terms in D. Therefore, the admissibility of D implies that $\sum Q_{l+2} + p_{l+2} \geq n - d - T$. Hence, by our construction, once we fix p_i and Q_i for $2 \leq i \leq l + 1$, then Q_{l+2} and p_{l+2} are uniquely determined, and satisfy the properties of l-constructibility. Note that p_{l+2} must satisfy the inequality $p_{l+2} \geq n - 2d$ since $T + \sum Q_{l+2} \leq d$ and $T + \sum Q_{l+2} + p_{l+2} \geq n - d$. By the same argument, each $p_i \geq n - 2d$. This proves that each p_i is chosen from the set of t_i's.

Similarly, Q_i and p_i are uniquely determined for $i = l + 2, i = l + 3$, and all the way up to $i = 2l + 1$. Once Q_{2l+1} and p_{2l+1} are chosen, we are left with k unselected terms for some $k \geq 0$. Then our only choice is to assign these k terms to Q_1. Thus, this assignment of p_i's and Q_i's must be unique, up to cyclic permutation. This completes the proof. □

In our example with $(n, d) = (89, 40)$, we have already determined p_i and Q_i for each $1 \leq i \leq 4$. By applying the above method, we see that $Q_5 = \{1\}$, $p_5 = 9$, and $Q_1 = \{1, 9\}$. We can readily verify that this representation of D into p_i's and Q_i's satisfies the properties of an l-constructible sequence. Thus, we have shown that every 2-constructible representation of D *must* be a cyclic permutation of

$$\underbrace{1, 9}_{Q_1}, \underbrace{20}_{p_1}, \underbrace{10}_{p_2}, \underbrace{19}_{p_3}, \underbrace{2, 9}_{Q_4}, \underbrace{9}_{p_4}, \underbrace{1}_{Q_5}, \underbrace{9}_{p_5}.$$

The next lemma shows that D is l-constructible for *only one* $l \geq 0$.

Lemma 3.13. *If D is l-constructible, then D is not l'-constructible, for any $l' \neq l$.*

Proof. Suppose that D is both l-constructible and l'-constructible. Without loss, suppose $l' < l$. Since D is l-constructible, we know that D can be expressed as

$$D = Q_1, p_1, Q_2, p_2, \ldots, Q_{2l+1}, p_{2l+1},$$

such that $\sum S \leq d$ if S contains at most l of the p_i's, and $\sum S \geq n - d$ otherwise.
If D is l'-constructible, then D can also be expressed as

$$D = Q_1', p_1', Q_2', p_2', \ldots, Q_{2l'+1}', p_{2l'+1}',$$

such that $\sum S \leq d$ if S contains at most l' of the p_i''s, and $\sum S \geq n - d$ otherwise.
For each $1 \leq j \leq 2l' + 1$, define X_j to be the subsequence

$$X_j = p_j', Q_{j+1}', p_{j+1}', \ldots, Q_{j+l'}', p_{j+l'}',$$

where the indices are reduced mod $(2l' + 1)$.
Since X_j contains exactly $l' + 1$ of the p_i''s, $\sum X_j \geq n - d$. This sequence X_j appears exactly as a subsequence of consecutive terms in $D = Q_1, p_1, Q_2, p_2, \ldots, Q_{2l+1}, p_{2l+1}$. Since $\sum X_j \geq n - d$, it follows that X_j must contain at least $(l + 1)$ of the p_i's, since D is l-constructible.
For each $1 \leq j \leq 2l' + 1$, define $\Gamma(Q_j')$ to be the number of p_i's that appear in Q_j', and define $\Gamma(p_j') = 1$ if $p_j' = p_i$ for some i, and $\Gamma(p_j') = 0$ otherwise.
Since X_j contains at least $l + 1$ of the p_i's, we must have

$$\Gamma(p_j') + \Gamma(Q_{j+1}') + \Gamma(p_{j+1}') + \cdots + \Gamma(Q_{j+l'}') + \Gamma(p_{j+l'}') \geq l + 1.$$

Summing over all $1 \leq j \leq 2l' + 1$, we have

$$l' \sum_{j=1}^{2l'+1} \Gamma(Q_j') + (l' + 1) \sum_{j=1}^{2l'+1} \Gamma(p_j') \geq (l + 1)(2l' + 1).$$

This identity follows because each $\Gamma(Q_j')$ is counted l' times and each $\Gamma(p_j')$ is counted $l' + 1$ times. This inequality can be rewritten as:

$$\sum_{j=1}^{2l'+1} \Gamma(Q_j') \geq \frac{(l + 1)(2l' + 1) - (l' + 1) \sum_{j=1}^{2l'+1} \Gamma(p_j')}{l'}.$$

For each $1 \leq j \leq 2l' + 1$, define Y_j to be the subsequence

$$Y_j = Q_j', p_j', Q_{j+1}', p_{j+1}', \ldots, Q_{j+l'-1}', p_{j+l'-1}', Q_{j+l'}',$$

where the indices are reduced mod $(2l' + 1)$.
Since Y_j contains exactly l' of the p_i''s, $\sum Y_j \leq d$. This sequence Y_j appears exactly as a subsequence of consecutive terms in

$$D = Q_1, p_1, Q_2, p_2, \ldots, Q_{2l+1}, p_{2l+1}.$$

Since $\sum Y_j \leq d$, it follows that Y_j contains at most l of the p_i's, since D is l-constructible. Therefore, we have

$$\Gamma(Q_j') + \Gamma(p_j') + \Gamma(Q_{j+1}') + \Gamma(p_{j+1}') + \cdots + \Gamma(p_{j+l'-1}') + \Gamma(Q_{j+l'}') \leq l.$$

Summing over all $1 \leq j \leq 2l' + 1$, we have

$$(l' + 1) \sum_{j=1}^{2l'+1} \Gamma(Q_j') + l' \sum_{j=1}^{2l'+1} \Gamma(p_j') \leq l(2l' + 1).$$

This inequality can be rewritten as

$$\sum_{j=1}^{2l'+1} \Gamma(Q_j') \leq \frac{l(2l'+1) - l' \sum_{j=1}^{2l'+1} \Gamma(p_j')}{l'+1}.$$

So now we have two inequalities in terms of $\sum_{j=1}^{2l'+1} \Gamma(Q_j')$ and $\sum_{j=1}^{2l'+1} \Gamma(p_j')$. From these two inequalities, we have

$$\frac{(l+1)(2l'+1) - (l'+1) \sum_{j=1}^{2l'+1} \Gamma(p_j')}{l'} \leq \frac{l(2l'+1) - l' \sum_{j=1}^{2l'+1} \Gamma(p_j')}{l'+1}$$

$$(l+1)(l'+1)(2l'+1) - (l'+1)^2 \sum_{j=1}^{2l'+1} \Gamma(p_j') \leq ll'(2l'+1) - l'^2 \sum_{j=1}^{2l'+1} \Gamma(p_j')$$

$$(2l'+1)(l+1)(l'+1) - (2l'+1)ll' \leq (2l'+1) \sum_{j=1}^{2l'+1} \Gamma(p_j')$$

$$\sum_{j=1}^{2l'+1} \Gamma(p_j') \geq ll' + l + l' + 1 - ll'$$

$$\sum_{j=1}^{2l'+1} \Gamma(p_j') = l + l' + 1$$

$$\sum_{j=1}^{2l'+1} \Gamma(p_j') > 2l' + 1 \quad \text{(since } l > l'\text{)}.$$

By the Pigeonhole Principle, we must have $\Gamma(p_j') > 1$ for some index j. However, each $\Gamma(p_j') \leq 1$ and this gives us our desired contradiction.

Therefore, we have shown that for any $l' \neq l$, D is not l'-constructible if D is l-constructible. $\quad \square$

We are finally ready to prove Theorem 3.9.

Proof. By the definition of an l-constructible sequence, every subsequence of consecutive terms has a sum outside the range $[d+1, n-d-1]$. Therefore, each l-constructible sequence is valid in B_n, for every $l \geq 0$. By Lemmas 3.12 and 3.13, we have shown that there is a bijection between the set of valid difference sequences of B_n and the union of all l-constructible sequences for $l \geq 0$. Every valid difference sequence D corresponds to a unique l-constructible sequence, for exactly one $l \geq 0$. To determine the number of valid difference sequences of B_n, it suffices to determine the number of l-constructible sequences for each $l \geq 0$, and then enumerate its union.

Let D be an l-constructible sequence, for some fixed $l \geq 0$. Thus, D is valid in B_n. By definition, any subsequence of consecutive terms containing l of the p_i's must sum to at most d.

Consider an l-constructible sequence $D = Q_1, p_1, Q_2, p_2, \ldots, Q_{2l+1}, p_{2l+1}$. We enumerate the number of all possible l-constructible sequences, for this fixed $l \geq 0$. We will show that each l-constructible sequence D must be generated in the following way:

(a) Choose $(a_1, a_2, \ldots, a_{2l+1})$ to be an ordered $(2l+1)$-tuple of non-negative integers with sum $k = (2l+1)d - ln$.
(b) Select $Q_1, Q_2, \ldots, Q_{2l+1}$ so that $\sum Q_j \leq a_{j+l+1}$ for each $1 \leq j \leq 2l+1$. Note that for $j \geq l+1$, the index $j+l+1$ is reduced mod $(2l+1)$.
(c) From this, each p_j is uniquely determined, and satisfies $p_j \geq n - 2d$.
(d) The sequence $D = Q_1, p_1, Q_2, p_2, \ldots, Q_{2l+1}, p_{2l+1}$ is l-constructible.

Each of these steps is easy to enumerate, and this will enable us to count the total number of l-constructible difference sequences.

Define $X_j = Q_j, p_j, \ldots, Q_{j+l-1}, p_{j+l-1}$ for each $1 \leq j \leq 2l+1$, where the indices are reduced mod $(2l+1)$. Since X_j contains l of the p_i's, $\sum X_j \leq d$. Let a_j be the integer for which $\sum X_j = d - a_j$. Then each $a_j \geq 0$.

Let $X_j' = X_j, Q_{l+j}$. Then $\sum X_j' \leq d$ because X_j' contains only l of the p_i's. Hence, $\sum X_j' = \sum X_j + \sum Q_{l+j} \leq d$, which implies that $\sum Q_{l+j} \leq a_j$. This is true for each j, so adding $l+1$ to both indices and reducing mod $(2l+1)$, we have $\sum Q_j \leq a_{j+l+1}$. Note that

$$\sum Q_j + p_j = n - \sum X_{j+1} - \sum X_{j+l+1}$$
$$= n - (d - a_{j+1}) - (d - a_{j+l+1})$$
$$= n - 2d + a_{j+1} + a_{j+l+1}.$$

Since $\sum Q_j \leq a_{j+l+1}$, it follows that $p_j \geq n-2d+a_{j+1} \geq n-2d$, which is consistent with the definition of l-constructibility.

Let $k = \sum a_j$. We have $\sum X_j = d - a_j$ for each j. Adding these $2l + 1$ sums, we have $\ln = (2l + 1)d - k$, or $k = (2l + 1)d - \ln \geq 0$. So k is fixed.

Since the a_j's are non-negative integers with sum k, a well-known combinatorial identity shows that there are $\binom{k+2l}{2l}$ ways to select the $(2l+1)$-tuple $(a_1, a_2, \ldots, a_{2l+1})$. For each of these $(2l+1)$-tuples, we select our Q_j's so that $\sum Q_j \leq a_{j+l+1}$ for each $1 \leq j \leq 2l + 1$. By Lemma 3.10, if our Q_j's have a *total* of t non-zero elements among them, then our selection of the Q_j's can be made in exactly $\binom{k}{t}$ ways.

This l-constructible sequence D will contain a total of $2l + t + 1$ terms, with t of them coming from the union of the Q_j's, and one for each of the $2l + 1$ p_i's. So there are $\binom{k+2l}{2l}\binom{k}{t}$ possible l-constructible sequences with $2l + t + 1$ terms. Therefore, there are this many valid difference sequences of B_n with $2l + t + 1$ terms. Note that some of these sequences are cyclic permutations of others, and we will take this into account when we determine the number of independent sets with $2l + t + 1$ vertices.

Let Ψ be the set of pairs (v, D), where v is a vertex of B_n and D is any of the $\binom{k+2l}{2l}\binom{k}{t}$ l-constructible sequences with $2l + t + 1$ elements. Each of the $n\binom{k+2l}{2l}\binom{k}{t}$ pairs in Ψ will correspond to an independent set I with $2l + t + 1$ vertices:

$$I = \{v, v + d_1, v + d_1 + d_2, \ldots, v + d_1 + d_2 + \ldots + d_{2l+t}\},$$

where the elements are reduced mod n and arranged in increasing order.

We now justify that each independent set I appears exactly $(2l + 1)$ times by this construction. The key insight is that each D is an l-constructible sequence, and hence has the following form:

$$D = Q_1, p_1, Q_2, p_2, \ldots, Q_{2l+1}, p_{2l+1}.$$

Therefore, there are exactly $(2l + 1)$ cyclic permutations of D so that it retains the form of an l-constructible sequence: for each cyclic permutation, the sequence begins with Q_i, for some $1 \leq i \leq 2l + 1$. Thus, we must divide the total number of independent sets by $(2l + 1)$, as each one is repeated this many times.

In other words, there are $\frac{n}{2l+1}\binom{k+2l}{2l}\binom{k}{t}$ independent sets with $2l + t + 1$ vertices.

Since this is true for each $l \geq 0$ and $0 \leq t \leq k = (2l + 1)d - \ln$, it follows that

$$
\begin{aligned}
I(B_n, x) &= 1 + \sum_{l \geq 0} \sum_{t=0}^{k} \frac{n}{2l + 1}\binom{k + 2l}{2l}\binom{k}{t} x^{2l+1+t} \\
&= 1 + \sum_{l \geq 0} \frac{n}{2l + 1}\binom{k + 2l}{2l} x^{2l+1} \sum_{t=0}^{k} \binom{k}{t} x^t \\
&= 1 + \sum_{l \geq 0} \frac{n}{2l + 1}\binom{k + 2l}{2l} x^{2l+1}(1 + x)^k \\
&= 1 + \sum_{l \geq 0} \frac{n}{2l + 1}\binom{(2l + 1)d - l(n - 2)}{2l} x^{2l+1}(1 + x)^{(2l+1)d - \ln}.
\end{aligned}
$$

Note that we require $k = (2l + 1)d - \ln \geq 0$ for there to be any independent sets. Thus, $l \leq \frac{d}{n-2d}$. Letting $r = n - 2d - 2$, we conclude that

$$I(C_{n, \{\pm d+1, \pm d+2, \ldots, \pm \lfloor \frac{n}{2} \rfloor\}}, x) = 1 + \sum_{l=0}^{\lfloor \frac{d}{r+2} \rfloor} \frac{n}{2l + 1}\binom{d - lr}{2l} x^{2l+1}(1 + x)^{d - l(r+2)}.$$

This concludes the proof of Theorem 3.9. $\square$

4. An application to music

The 12-semitone music scale consists of the pitch classes C, $C^\#$, D, $D^\#$, E, F, $F^\#$, G, $G^\#$, A, $A^\#$, and B. Each note is identified with its pitch class (i.e., each C refers to the same note, regardless of its octave). These 'pitch classes' are the musical analogue of equivalence classes.

Suppose we want to play a chord consisting of $k \geq 3$ different pitch classes from this scale. Clearly, the number of different possibilities is $\binom{12}{k}$. But if we were to introduce *forbidden intervals* and ask for the number of chords we could play with this restriction, then we can answer this problem using independence polynomials. In particular, if the forbidden intervals

correspond to pitch classes that are close together (and hence, dissonant), we show that this problem can be answered from the independence polynomial $I(C_n^d, x)$.

As a simple example, suppose that we are forbidden to include any chord with two pitch classes separated by a semitone or tone (for example, C and $C^{\#}$, or G and A). In other words, if we were to draw a graph with these 12 pitch classes as our vertices, we would require every pair of pitch classes to be separated by a distance of at least three (i.e., a minor third), to avoid any semitones or tones. Now we can ask how many possible chords can be played with this given restriction.

$$I(C_{12}^2, x).$$

Mathematically, this is equivalent to the problem of evaluating the independence polynomial $I(C_{12}^2, x)$, and then substituting $x = 1$ to determine our answer. In other words, every possible chord is some independent set of size at least 3 in the circulant C_{12}^2, since each pair of pitch classes in an independent set is separated by at least a minor third (three semitones). By Theorem 3.3,

$$I(C_{12}^2, x) = \sum_{k=0}^{4} \frac{12}{12 - 2k} \binom{12 - 2k}{k} x^k = 1 + 12x + 42x^2 + 40x^3 + 3x^4.$$

Thus, $I(C_{12}^2, 1) = 98$. We conclude that there are $98 - (1 + 12 + 42) = 43$ possible chords that can be played, excluding the 55 trivial 'chords' of less than three pitch classes (corresponding to the 55 independent sets of size at most 2 in C_{12}^2).

We can generalize the 12-semitone octave to the n-semitone octave. As in the 12-semitone octave, the n-semitone octave is divided into n equally tempered tones, each formed by multiplying the frequency by $2^{\frac{1}{n}}$. Musicians refer to this as the n-tet scale (where 'tet' is an acronym of Tone Equal-Tempered); see, for example, [29]. Traditional Thai instruments are tuned to a scale that is approximately 7-tet, and various composers have written music in n-tet scales for other values of n (one common one, 19-tet, is well-suited as the ratio of $3/2$, a *perfect fifth*, can be approximated very closely).

In an n-tet scale the ratio between any two semitones is constant. Since notes with 'close' frequencies sound dissonant when played together, we can require that no chord include two pitch classes separated by d semitones or less, for some integer $d \geq 1$. Let $f(n, d)$ be the number of possible non-trivial chords that can be played with this restriction. By Theorem 3.3, the answer is simply

$$f(n, d) = \sum_{k=3}^{\lfloor \frac{n}{d+1} \rfloor} \frac{n}{n - dk} \binom{n - dk}{k},$$

which we derive from evaluating $I(C_n^d, x)$ at $x = 1$, and then subtracting the number of trivial chords with less than three pitch classes.

This gives us the formula for the number of 'optimal' chords. We could also determine the number of 'least-optimal' chords, where each pair of pitch classes is separated by at most d semitones. In other words, each pair of notes sound dissonant when played together, i.e., the selection of the pitch classes is the worst possible. Let $g(n, d)$ be the number of possible non-trivial chords that can be played with this 'reverse' restriction, where $r = n - 2d - 2 \geq 0$. By Theorem 3.9, the answer is simply

$$g(n, d) = 1 + \sum_{k=3}^{\lfloor \frac{d}{r+2} \rfloor} \frac{n}{2k + 1} \binom{d - kr}{2k} 2^{d - k(r+2)}.$$

We remark that our counting method also extends to equally tempered subdivisions of stretched and shrunken octaves, that is, those scales where the ratio of frequency of the top and bottom frequencies is larger or smaller than 2, respectively.

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