On Circulants Uniquely Characterized by their Independence Polynomials

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Abstract

In [18], Farrell and Whitehead investigate circulant graphs that are uniquely characterized by their matching and chromatic polynomials (i.e., graphs that are "matching unique" and "chromatic unique"). They develop a partial classification theorem, by finding all matching unique and chromatic unique circulants on n vertices, for each $n \leq 8$. In this paper, we explore circulant graphs that are uniquely characterized by their *independence polynomials*. We obtain a full classification theorem by proving that a circulant is independence unique iff it is the disjoint union of isomorphic complete graphs.

Keywords: circulant graph, matching polynomial, chromatic polynomial, independence polynomial, threshold graphs, spider graphs.

1 Introduction

Graph polynomials are an important topic of interest to many combinatorialists. The coefficients of a graph polynomial encode various combinatorial properties of a graph, such as the number of independent sets or the number of matchings. There are several graph polynomials that are active areas of combinatorial research, such as chromatic polynomials, matching polynomials, reliability polynomials, Tutte polynomials, and independence polynomials.

Given an arbitrary graph G on n vertices, we can compute a graph polynomial by enumerating the number of occurrences of a particular property. We see this in the definition of the following three graph polynomials.

Definition 1.1 *Let G be a graph. In an* independent set *of k vertices, no two vertices are adjacent in G. In a* matching *of k edges, no two edges share a common vertex. In a* proper colouring *of G with x colours, no two adjacent vertices receive the same colour.*

The **independence polynomial** $I(G,x)$ *is* $\sum_{k\geq 0} i_k x^k$*, where* i_k *is the number of independent sets of cardinality* k *in* G*.*

The **matching polynomial** $M(G,x)$ *is* $\sum_{k\geq 0}(-1)^k m_k x^{n-2k}$*, where* m_k *is the number of matchings in* G *with exactly* k *edges.*

The **chromatic polynomial** $\pi(G,x)$ *is the function that gives the number of proper colourings of the vertices of* G *using* x *colours.*

For example, we compute each of these three graph polynomials for the triangle K_3 and the 6-cycle C_6.

$$\begin{aligned}
I(K_3,x) &= 3x+1\\
M(K_3,x) &= x^3-3x\\
\pi(K_3,x) &= x(x-1)(x-2) = x^3-3x^2+2x
\end{aligned}$$

$$\begin{aligned}
I(C_6,x) &= 2x^3+9x^2+6x+1\\
M(C_6,x) &= x^6-6x^4+9x^2-2\\
\pi(C_6,x) &= x^6-6x^5+15x^4-20x^3+15x^2-5x
\end{aligned}$$

Every graph polynomial is well-defined: for example, a graph cannot have two different independence polynomials. However, does the converse hold? Are there any graph polynomials that are unique to that graph? In this paper, we investigate graphs that have a unique independence polynomial, and determine a necessary and sufficient condition for when a *circulant* graph has a unique independence polynomial.

For notational convenience, we introduce the following definition, which holds for every P-polynomial (where P is replaced by "chromatic", "matching", "independence", etc.)

Definition 1.2 *An graph* G *is* **P-unique** *if* G *is uniquely characterized by its* P*-polynomial. In other words,* $P(G,x)=P(H,x)$ *implies that* G *is isomorphic to* H*.*

Much work has been done to characterize graphs that are chromatically unique [6, 8, 16, 23, 27, 30, 33], in addition to graphs that are Tutte unique [13] and reliability unique [7]. Since the problem of independence unique graphs was first posed in [21], very few results have been found [26]. Some work has been conducted on classifying independence unique graphs for *spider graphs* [25] and *threshold graphs* [32]. However, other than these two specific families of graphs, not much is known. In the recent survey paper on independence polynomials [26], these are the only two families of graphs that are discussed.

An independence polynomial $I(G,x)$ of the form $1+nx+\ldots$ corresponds to a graph G on n vertices. To prove that G is independence unique, we must theoretically examine all graphs on n vertices and determine their independence polynomials. For small values of n, the computation is trivial. However, for an arbitrary graph, it is NP-hard [19] to compute the independence polynomial. We remark that although the following recurrence relation is computationally inefficient, any $I(G, x)$ can be computed with this result.

Theorem 1.3 ([20]) *For any vertex v,*

$$I(G, x) = I(G - v, x) + x \cdot I(G - N[v], x),$$

where the closed neighbourhood $N[v]$ is the set $\{u : \ u = v \text{ or } uv \in E\}$.

We now define the *circulant graph* $C_{n,S}$.

Definition 1.4 *Given a set $S \subseteq \{1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor\}$, the* **circulant graph** $C_{n,S}$ *is the graph with vertex set $V(G) = \mathbb{Z}_n$, and edge set*

$$E(G) = \{uv : |u - v|_n \in S\},$$

where $|x|_n = \min\{|x|, n - |x|\}$ is the **circular distance** *modulo n.*

The generating set S will always refer to a subset of $\{1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor\}$. In the literature, S is also referred to as the *connection set* [2, 14].

For example, here are the circulants $C_{9,\{1,2\}}$ and $C_{9,\{3,4\}}$.

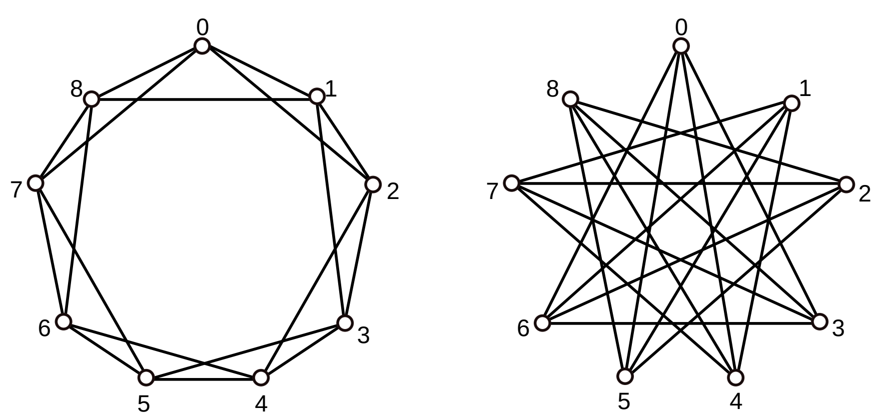

Figure 1: The circulant graphs $C_{9,\{1,2\}}$ and $C_{9,\{3,4\}}$.

Note that the circulant $C_{n,\{1,2,3,\ldots,\lfloor \frac{n}{2} \rfloor\}}$ is simply the complete graph K_n. Also, $C_{n,\{1\}}$ is just the cycle C_n. In general, $C_{n,\{d\}} \simeq C_n$ for any d with $\gcd(d, n) = 1$.

Circulant graphs have been investigated in fields outside of graph theory. For example, for geometers, circulant graphs are known as *star polygons* [10]. Circulants have been used to solve problems in group theory, as shown in [2], as well as number theory and analysis [11]. They are well-studied in network theory, as they model practical data connection networks [3, 22]. Circulant graphs (and circulant matrices) have important applications to the theory of designs and error-correcting codes [31].

In [18], Farrell and Whitehead investigate circulants that are chromatically unique and matching unique. They prove that of the 30 non-isomorphic circulants of order at most eight, 23 are chromatically unique, with the seven exceptions being $C_{4,\{2\}}$, $C_{6,\{2\}}$, $C_{6,\{3\}}$, $C_{8,\{2\}}$, $C_{8,\{4\}}$, $C_{8,\{2,4\}}$, and $C_{8,\{1,3,4\}}$. Then they prove that each of these seven circulants is matching unique, proving that every circulant on $n \leq 8$ vertices is either chromatically unique or matching unique (or both). While they are unable to verify this conjecture for any $n \geq 9$, they conjecture that this result holds for all n.

Their analysis motivates the equivalent problem for independence polynomials: can we determine some independence unique circulants? If so, can we characterize them? In this section, we provide a full answer to the uniqueness problem for independence polynomials: we prove that a circulant is uniquely characterized by its independence polynomial iff it is the disjoint union of isomorphic complete graphs (e.g. $C_{8,\{1,2,3,4\}}$ and $C_{24,\{3,6,9,12\}}$). In other words, many circulants appear to be chromatic and matching unique, but circulants are independence unique only in a handful of cases. In fact, we will prove that there are exactly $\phi(n)$ circulants on n vertices, where $\phi(n)$ denotes the number of positive divisors of n.

Some simple examples of independence unique graphs include K_n and $\overline{K_n}$. To give an example of a graph G that is not independence unique, consider the complement of a tree. For any tree H on n vertices, let $G = \overline{H}$. Then,

$$I(\overline{G}, x) = 1 + nx + (n-1)x^2.$$

So any complement of an n-vertex tree has the same independence polynomial.

It is shown in [15] that two non-isomorphic trees can have the same independence polynomial, as illustrated in Figure 2.

Figure 2: Two trees with the same independence polynomial.

In the above example,

$$I(T_1, x) = I(T_2, x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 12x^5 + x^6.$$

Based on this analysis, we may ask if two non-isomorphic circulants can have the same independence polynomial. We will answer that question in Section 3. First, we establish our characterization theorem of all independence unique circulants.

2 Main Theorem

We will now give a complete characterization of independence unique circulants, where we prove the surprising result that a circulant G is independence unique iff G is the disjoint union of isomorphic complete graphs (see Theorem 2.5). This result will follow quickly from the following key theorem.

Theorem 2.1 *Let $G = C_{n,S}$ be a connected circulant graph. Then, G is independence unique iff $G \simeq K_n$.*

We conclude that circulants are not rich in independence unique graphs, even though they are rich in chromatically unique and matching unique graphs. To prove Theorem 2.1, we first require the following definition and lemma.

Definition 2.2 *Let $G = C_{n,S}$ be a circulant graph. Define*

$$S' = \{x : \ |x|_n \in S\} \cup \{0\}.$$

Note that $S' = N_G[0]$, the closed neighbourhood of vertex 0 in G.

For each $i \in S'$, the graph $\boldsymbol{H_i}$ is formed by taking $G - \{0\}$, creating a new vertex u, and then joining u to every vertex $y \in V(G - \{0\})$ for which $y = i + r \pmod{n}$ for some $r \in S'$.

For example, let $G = C_{8,\{3,4\}}$, and $i = 3$. The graphs G and H_3 are illustrated in Figure 3.

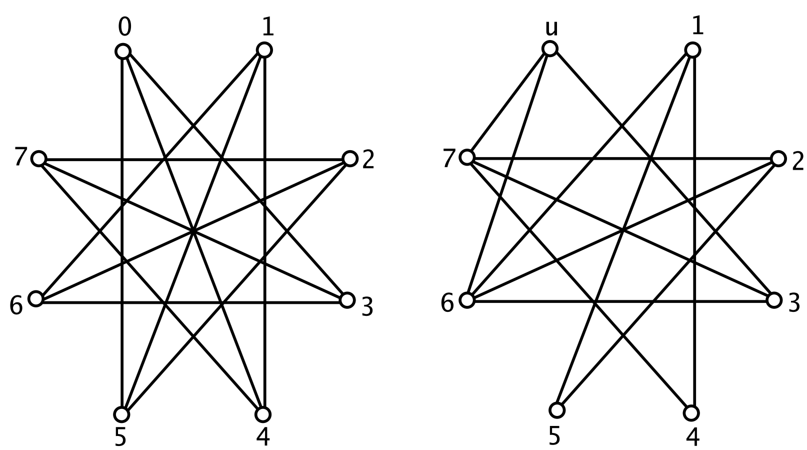

Figure 3: The graphs $G = C_{8,\{3,4\}}$ and the corresponding graph H_3.

Lemma 2.3 *For each $i \in S'$, $I(H_i, x) = I(G, x)$.*

Proof It is clear that $H_i - \{u\} = G - \{0\}$. Note that $x \in V(G - N_G[0])$ iff $x \notin S'$, and $y \in V(H_i - N_{H_i}[u])$ iff $y - i \pmod{n} \notin S'$.

Letting $\phi(x) = x + i \pmod{n}$, we see that ϕ is an isomorphism from $G - N_G[0]$ to $H_i - N_{H_i}[u]$. Therefore, $I(G - N_G[0], x) = I(H_i - N_{H_i}[u], x)$. By Theorem 1.3,

$$\begin{aligned} I(H_i, x) &= I(H_i - \{u\}, x) + x \cdot I(H_i - N_{H_i}[u], x) \\ &= I(G - \{0\}, x) + x \cdot I(G - N_G[0], x) \\ &= I(G, x). \end{aligned}$$

Therefore, we conclude that $I(H_i, x) = I(G, x)$. □

By this lemma, G is not independence unique whenever we can find $i \in S$ such that $H_i \not\simeq G$. So in the above example, $G = C_{8,\{3,4\}}$ is not independence unique.

We require one additional result, which is a straightforward observation.

Proposition 2.4 ([5]) *Suppose that the circulant graph $C_{n,S}$ has the generating set $S = \{s_1, s_2, \ldots, s_m\}$. Then $C_{n,S}$ is connected iff*

$$d = \gcd(n, s_1, s_2, \ldots, s_m) = 1.$$

We now prove Theorem 2.1.

Proof If $G \simeq K_n$, then $I(G, x) = 1 + nx$. Clearly G is independence unique, as any graph H with $I(H, x) = 1 + nx$ must have n vertices and satisfy $\alpha(H) = 1$.

Now suppose that G is independence unique, where G is a connected circulant. By Lemma 2.3, the graph H_i satisfies $I(H_i, x) = I(G, x)$ for all $i \in S'$. Since G is independence unique, each H_i must be isomorphic to G.

Since G is a circulant, G must be r-regular, for some r. Then the degree of each vertex in H_i must also be r. By definition, $G - \{0\} = H_i - \{u\}$. It follows that 0 and u must connect to the same set of vertices in $G - \{0\}$ and $H_i - \{u\}$, respectively, as otherwise $\deg_{H_i}(w) \neq \deg_G(w) = r$ for some vertex w. By definition of H_i, this implies that $x \in S'$ iff $x + i \pmod{n} \in S'$. This implication is true for all $i \in S'$.

Let i be the smallest non-zero element of S'. Then, $ki \pmod{n} \in S'$ for all $k \in \mathbb{N}$. By the Euclidean Algorithm, there exists an integer k such that $ki \pmod{n} = \gcd(i, n) \in S'$. By the minimality of i, this implies that $\gcd(i, n) = i$, so $i | n$. If $i = 1$, then $S' = \mathbb{Z}_n$, which implies that $S = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, i.e., $G \simeq K_n$.

Now let us assume that $i > 1$. If S' only contains multiples of i, then $G = C_{n,S}$ is disconnected by Proposition 2.4, which contradicts our given assumption that G is connected. So we can assume that there is an element $j \in S'$ with $d = \gcd(i, j) < i$. Since $x \in S'$ implies that $x + i \pmod{n} \in S'$ and $x + j \pmod{n} \in S'$, it follows that $pi + qj \pmod{n} \in S'$ for all pairs of positive integers (p, q). By the Euclidean algorithm, there exists a pair (p, q) for which $pi + qj \pmod{n} = d = \gcd(i, j) < i$, which contradicts the minimality of i.

We conclude that if $G = C_{n,S}$ is a connected circulant graph that is independence unique, then $G \simeq K_n$. This completes the proof. □

As a direct result of Theorem 2.1, we now establish that a circulant $G = C_{n,S}$ is independence unique iff G is the disjoint union of isomorphic complete graphs.

Theorem 2.5 *The circulant graph $G = C_{n,S}$ is independence unique iff $n = dk$ and $S = \{d, 2d, 3d, \ldots, \lfloor \frac{k}{2} \rfloor d\}$ for some positive integers k and d.*

Proof Let $G = C_{n,S}$ be independence unique, and let $S = \{s_1, s_2, \ldots, s_m\}$. By Proposition 2.4, $C_{n,S}$ is connected iff $d = \gcd(n, s_1, s_2, \ldots, s_m) = 1$. If $d = 1$, then $G = K_n$ by Theorem 2.1. So suppose $d > 1$, and let $n = dk$. Then G is the disjoint union of d isomorphic copies of $G' = C_{n',S'}$, where $n' = \frac{n}{d}$ and $s'_i = \frac{s_i}{d}$ for each $1 \leq i \leq m$. If $G' \not\simeq K_{n'}$, then there exists a graph H' not isomorphic to G' for which $I(G', x) = I(H', x)$. Letting H be the disjoint union of d copies of H', we have $I(G, x) = (I(G', x))^d = (I(H', x))^d = I(H, x)$. In other words, if $G = C_{n,S}$ is independence unique, we must have $G' = K_k$, i.e., $n' = k$ and $S' = \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$. The desired conclusion follows. □

We now enumerate the number of independence unique circulants on n vertices, for each integer $n \geq 1$. From the above theorem, this question is easily answered.

Corollary 2.6 *Define $\phi(n)$ to be the number of positive divisors of n. Then there are $\phi(n)$ independence unique circulants on n vertices, for each $n \geq 1$.*

Proof From Theorem 2.5, $G = C_{n,S}$ is independence unique iff $n = dk$ for some ordered pair $(d, k) = (d, \frac{n}{d})$. In this case, the generating set S is uniquely defined. Therefore, exactly one independence unique circulant exists for each $d|n$. The desired conclusion follows. □

3 Further Exploration

We have now proven that other than disjoint unions of isomorphic complete graphs, circulant graphs are not independence unique. We proved this by constructing a non-circulant graph H with $I(G, x) = I(H, x)$. Let us explore this concept further.

If G and H are both restricted to the family of circulants: must they have different independence polynomials? A variation of this question is posed in [32], where it is shown that if Γ is the family of *threshold graphs*, then $I(G, x) \neq I(H, x)$ whenever G and H are non-isomorphic graphs in Γ. If Γ is the family of well-covered *spider graphs* (i.e., trees having at most one vertex of degree ≥ 3), then it is known [25] that $I(G,x) \neq I(H,x)$ for all $G, H \in \Gamma$ with $G \not\simeq H$. In other words, every graph in Γ has a unique independence polynomial within these two families. This motivates the following question, for the family of circulant graphs.

Problem 3.1 *Let G and H be circulants. If $I(G,x) = I(H,x)$, then must this imply that $G \simeq H$?*

We prove that the answer is no. The following is the minimum counterexample (i.e., counterexample with the fewest number of vertices).

Proposition 3.2 *Let $G = C_{8,\{1,2,4\}}$ and $H = C_{8,\{1,3,4\}}$. Then, $I(G,x) = I(H,x)$ but $G \not\simeq H$.*

Proof It is easily checked that $I(G,x) = I(H,x) = 1 + 8x + 8x^2$. To prove that $G \not\simeq H$, it suffices to show that $\overline{G} \not\simeq \overline{H}$. But this is clear, since $\overline{G} = C_{8,\{3\}}$ is isomorphic to C_8, and $\overline{H} = C_{8,\{2\}}$ is isomorphic to two disjoint copies of C_4. □

To give another example, $G = C_{13,\{1,2,4\}}$ and $H = C_{13,\{1,3,4\}}$ are graphs satisfying $I(G,x) = I(H,x)$ and $G \not\simeq H$. The non-isomorphism of G and H is verified by noting that the 5-wheel W_5 is an induced subgraph of G, but not of H.

Proposition 3.2 can also be proved by comparing the sets of eigenvalues of G and H, and showing that they are different. To compute the eigenvalues of a graph, we find its adjacency matrix A, and then the eigenvalues correspond to all scalars λ such that $Ax = \lambda x$ for some non-zero vector x.

In a circulant $G = C_{n,S}$, each row of the adjacency matrix A is a cyclic permutation of every other row. Let $a = [a_0, a_1, \ldots, a_{n-1}]$ be the first row of A, where $a_i = a_{n-i} = 1$ iff $i \in S$. Several papers and books have been written on circulant matrices and their properties [4, 9, 12, 24, 31]. In all of these works, the eigenvalues of these matrices are studied. Here is the formula for the eigenvalues of $C_{n,S}$.

Theorem 3.3 ([9]) *If n is odd, the eigenvalues of G are*

$$\lambda_0 = \sum_{k=1}^{(n-1)/2} 2a_k, \quad \lambda_j = \lambda_{n-j} = \sum_{k=1}^{(n-1)/2} 2a_k \cos\left(\frac{2jk\pi}{n}\right) \quad \text{for } 1 \leq j \leq \frac{n-1}{2}.$$

If n is even, then for all $1 \leq j \leq \frac{n}{2}$,

$$\lambda_0 = a_{n/2} + \sum_{k=1}^{n/2-1} 2a_k, \quad \lambda_j = \lambda_{n-j} = a_{n/2}\cos(j\pi) + \sum_{k=1}^{n/2-1} 2a_k \cos\left(\frac{2jk\pi}{n}\right).$$

We can manually verify that the set of eigenvalues of $G = C_{8,\{1,2,4\}}$ is different from those of $H = C_{8,\{1,3,4\}}$. Hence, we must have $G \not\simeq H$. □

Therefore, we conclude that there are pairs of non-isomorphic circulants that have the same independence polynomial. There are several techniques to verify that two circulants are (not) isomorphic. For example, the techniques discussed in Proposition 3.2 and Theorem 3.3 are straightforward, but tedious. Is there a simpler method to determine whether two circulants $C_{n,S}$ and $C_{n,T}$ are isomorphic?

The following lemma provides a simple sufficient condition for isomorphism.

Lemma 3.4 ([28]) *Let S be any subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. For each integer $r \geq 1$, define $rS = \{|rs|_n : s \in S\}$. If $T = rS$ for some integer r (with $\gcd(r, n) = 1$), then T is a* **multiplier** *of S.*

If $T = rS$ is a multiplier of S, then $C_{n,S} \simeq C_{n,T}$.

In [1], Ádám conjectured that Lemma 3.4 is also a necessary condition. This was later disproved [17]. To give one counterexample, $C_{16,\{1,2,7\}} \simeq C_{16,\{2,3,5\}}$, yet there is no r for which $\{2, 3, 5\} \equiv \{r, 2r, 7r\} \pmod{16}$. It is known that the conjecture is false if n is divisible by 8 or is the square of an odd prime [28]. However, Ádám's conjecture is true whenever n is square-free, i.e., there is no integer $d > 1$ with $d^2 | n$.

Theorem 3.5 ([28]) *If n is square-free, then $C_{n,S} \simeq C_{n,T}$ iff there exists an integer r with $\gcd(r, n) = 1$ such that $T = rS$.*

By Theorem 3.5, we immediately have another proof that $C_{13,\{1,2,4\}} \not\simeq C_{13,\{1,3,4\}}$.

A complete solution to the isomorphism problem for circulant graphs was recently given by Muzychuk [29]. The results are developed in the context of Schur rings, and an efficient algorithm is given for recognizing isomorphism between two circulant graphs.

4 Conclusion

We conclude the paper with three open problems.

Problem 4.1 *We proved that a circulant graph is independence unique iff it is the disjoint union of isomorphic complete graphs. Determine if a full characterization of independence unique graphs can be found for the set of all graphs.*

Problem 4.2 *Extend the analysis of Farrell and Whitehead [18] by classifying all circulant graphs that are chromatic unique.*

Problem 4.3 *Extend the analysis of Farrell and Whitehead [18] by classifying all circulant graphs that are matching unique.*

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