Inside Microsoft Research

- Computer Magazine, January 1998, pages 51-58.
- NP-hard problems and Theory
  - investigation of critical transition points
  - applications: cryptography and networking

Physics Critical Points

In physics, you are exploring very complex states containing millions of molecules and looking at average behavior versus radical change. For example, if you vary a parameter, such as temperature, that average complex behavior will undergo a radical change -- from liquid to gas, for example, at a critical point.

Computer Science Critical Points

- In computer science, there might well be an analogous transition point at which an easy problem suddenly becomes hard to solve, or vice versa.

Yes, of course!

- Ernst Specker and Karl Lieberherr developed a theory of P-optimal approximation algorithms which provides an infinite family of transition points (1976-1983).
- Started with the Golden Ratio result published in the Journal of the ACM.

Simple example

One-Out-Of-Three Problem(f):
Given a constraint system (a bag of constraints) expressed in n variables which may assume only the values 0 or 1, find an
assignment to the n variables which satisfies
at least the fraction f of the constraints.

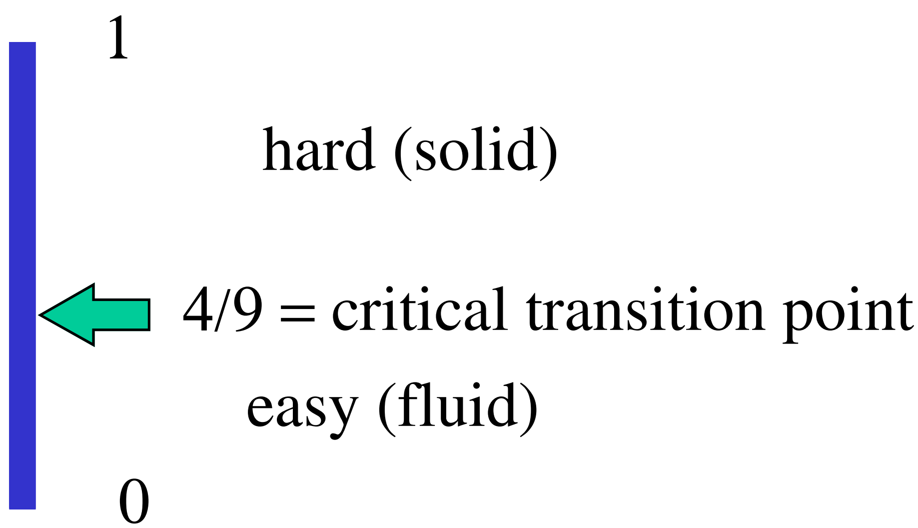
Example: Constraints are of the form x+y+z=1.

$$
\begin{array}{lllll}
x1 + x2 + x3 & & = 1 & & \\
x1 + x2 + & + x4 & = 1 & & \text{can satisfy } 3/4 \\
x1 + & x3 + x4 & = 1 & & \\
& x1 + x3 + x4 & = 1 & &
\end{array}
$$

Simple example

One-Out-Of-Three Problem(f):

For f <= 4/9: problem has polynomial solution
For f = 4/9 + ε: problem is NP-complete, ε>0.



Example

$$
\begin{aligned}
x1 + x2 + x3 \qquad\quad &= 1 \\
x1 + x2 + \quad\;\; + x4 &= 1 \\
x1 + \qquad\;\; x3 + x4 &= 1 \\
x1 + x3 + x4 &= 1 \\
x1 + x2 + \qquad\qquad + x5 &= 1 \\
x1 + \qquad\;\; x3 + \qquad\;\; x5 &= 1 \\
x2 + x3 + \qquad\;\; x5 &= 1
\end{aligned}
$$

can satisfy 6/7

Forget about computation ...

- Focus on purely mathematical question first
- Algorithmic solution will follow
- Mathematical question: Given a constraint system S, which fraction of the constraints can always be satisfied by some (0,1) assignment? In which constraint systems is it impossible to satisfy many constraints? What is the worst-case?

min max problem

p(S,I) = fraction of satisfied constraints in system S by assignment I

min	max	p(S,I)
all constraint systems S	all (0,1) assignments I	

= 4/9

min max problem

p(S,I,n) = fraction of satisfied constraints in system S by assignment I

lim	min	max	p(S,I,n)
n to infinity	all constraint systems S with n variables	all (0,1) assignments I to n variables	

= 4/9

Problem reductions are the key

- Solution to simpler problem implies solution to original problem.

min max problem

p(S,I,n) = fraction of satisfied constraints in system S by assignment I with n vars.

lim	min	max	p(S,I,n)
n to infinity	all SYMMETRIC constraint systems S with n variables	all (0,1) assignments I to n variables	

= 4/9

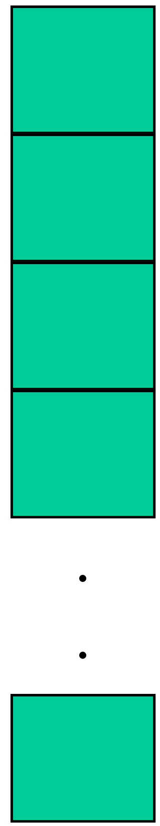
Reduction achieved

- Instead of minimizing over all constraint systems it is sufficient to minimize over the symmetric constraint systems.

Reduction

- Symmetric case is the worst-case: If in a symmetric constraint system the fraction f of constraints can be satisfied, then in any constraint system the fraction f can be satisfied.
- In a symmetric constraint system, the fraction 4/9 of constraints can be satisfied.

Symmetric the worst-case



n variables
n! permutations

If in the big system the
fraction f is satisfied,
then there must
be a least one small system
where the fraction f is satisfied

min max problem

p(S,I,n) = fraction of satisfied constraints in system S by assignment I

lim	min	max	p(S,I,n)
n to infinity	all SYMMETRIC constraint systems S with n variables	all (0,1) assignments I to n variables where the first k variables are set to 1	

= 4/9

Why correct reduction?

- In a symmetric constraint system it only matters how many of the variables are set to 1.

max problem

maximize $3x(1-x)^2$ for x between 0 and 1

$= 4/9$

Simple combinatorics

n variables, k set to true

k binomial(n-k,2) / binomial(n,3) will be satisfied
find maximum k

set $x=k/n$: maximize $3x(1-x)^2$
max at $x = 1/3$. value = $4/9$

Make it efficiently constructive

- There is a polynomial algorithm which finds an assignment satisfying at least the fraction 4/9 of constraints.
- Idea: Decide whether $x=1$ or $x=0$ produces better result in reduced symmetric formula.

Sketch of algorithm MAXMEAN

- $mean_k(S)$ is the average fraction of satisfied constraints in S among all assignments having exactly k ones.
- Compute k such that $mean_k(S)$ is max.
- for all variables x in S do

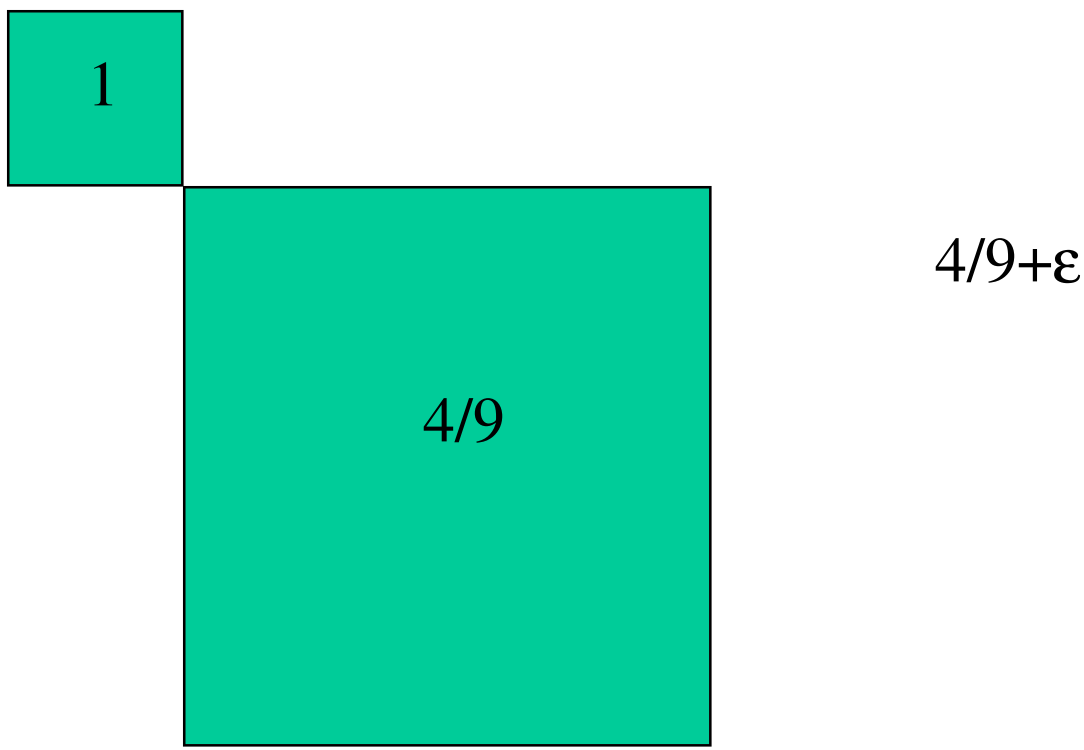
if $mean_{k-1}(S_{x=1}) > mean_k(S_{x=0})$ then

$j[x]=1;\ k=k-1; S=S_{x=1}$ else $j[x]=0;\ S=S_{x=0}$

More is hard

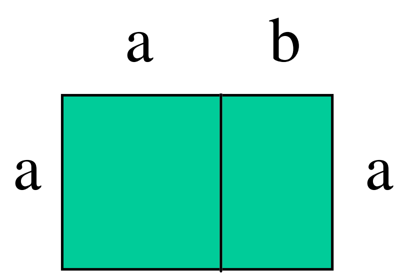
- The set of constraint systems which have an assignment satisfying at least the fraction $4/9+\varepsilon$ of the constraints is NP-complete. Idea: the set of constraint systems in which all constraints can be satisfied is NP-complete.

Reduction idea



Generalizations

- Use any finite set of relations to define constraint systems; can even consider forbidden substructures
- Satisfiability: 1/2
- Satisfiability with any pair satisfiable: Golden Ratio = (sqrt(5) - 1)/2



a/b=(a+b)/a, 1/x=1+x, x= Golden Ratio
Frequent ratio in Greek architecture

More examples

- R_j is the relation of rank r which holds if exactly j of the r variables are set to 1. Let $P=\{R_0, \ldots, R_r\}$.
- Critical point is: $1/(r+1)$.

Example: unpublished

- Let $F(p,q)$ be the following class of propositional formulas in conjuctive normal form: Each clause in a formula in $F(p,q)$ contains at least p positive or q negative literals $(p,q >= 1)$. Let a be the solution of $(1-x)^p = x^q$ in (0,1) and let $t(p,q) = 1-a^q$
- Critical point is: $t(p,q)$.

References

- Lieberherr/Specker, Journal of the ACM
- Lieberherr, Journal of Algorithms
- see my resume on my home page