4 October 2016
Analysis I
Paul E. Hand
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Coverpage to Pledged HW 6

Time limit: 3 hours. You may not use your books, your homeworks, your notes, or any electronics during the exam. Please write the start and finish times on your paper. To receive full credit, you must name all major theorems and state definitions used in your arguments. All counter examples must be accompanied by a proof. You may cite results from class and well-known theorems.

This homework is pledged. On the first page, please write your signature and the Rice University pledge: "On my honor, I have neither given nor received any unauthorized aid on this homework."

Due: Tuesday, 18 October 2016 at the beginning of class.

[The exam is on the next page]

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1. (10 points) Is $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ uniformly continuous over $\mathbb{R}$? Prove it.

2. (10 points) Let $x^{(\infty)} \in \mathbb{R}^n$. Suppose $x^{(k)} \in \mathbb{R}^n$ for $k \in \mathbb{N}$. Directly show that the following are equivalent:

   (i) $x^{(k)}$ converges to $x^{(\infty)}$ as $k \to \infty$ with respect to the ℓ_2 norm.

   (ii) $x^{(k)}$ converges to $x^{(\infty)}$ as $k \to \infty$ with respect to the sup norm.

   Do not appeal to the equivalence of norms on finite dimensional spaces. If you wish, you may begin by directly proving these two norms are equivalent.

3. (15 points) Let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of the rationals in $[0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ be such that $f(q_n) = \frac{1}{n}$ for all $n \in \mathbb{N}$ and $f(x) = 0$ for any irrational x. Is f Riemann integrable? Prove it.

4. Let V be the set of all real sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\sum_{n=1}^{\infty} \frac{x_n^2}{n} < \infty$.

   (a) (5 points) Find an element of V that is not in ℓ_2.

   (b) (10 points) Prove that $\|\cdot\| : \{x_n\}_{n=1}^{\infty} \mapsto \sqrt{\sum_{n=1}^{\infty} \frac{x_n^2}{n}}$ is a norm on V.