Focusing in logic programming

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1 Proof search in sequent calculus

Here are the rules for a standard intuitionistic sequent calculus with implication ($\supset$) and conjunction ($\wedge$).

$$\frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \supset L \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \supset R$$

$$\frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R$$

$$\frac{}{\Gamma, p \Rightarrow p} \mathsf{id}$$

1.1 Forward and backward chaining

Consider the following sequent:

$$p, p \supset q, q \supset r, s \supset r \Rightarrow r$$

By inspection, the only way to make progress on proving this sequent is to choose one of the implications on the left to apply $\supset L$ to. But which one? We could notice that the consequents of both $q \supset r$ and $s \supset r$ match our goal, and choose one of them on that basis. On the other hand, we could notice that we already have the antecedent of $p \supset q$ in our context, and choose it on that basis. The former choice corresponds to a *backward-chaining* (sometimes called "top down") strategy, and the latter corresponds to a *forward-chaining* (sometimes called "bottom up") strategy.

1.2 Invertible rules

Any time we need to make a choice like this ("which rule should I apply?"), we're making a decision that could affect whether or not our proof will succeed. We might have to backtrack over that choice if our proof fails: for example, choosing $s \supset r$ will require us to prove its antecedent s, which we can't. On the other hand, the other inference rules apart from $\supset L$ don't have this property:

we can "eagerly" apply them without worrying about whether we might have to backtrack. Let's call these rules *invertible*, and the rules that represent choices that might fail *non-invertible*.

Invertible rules:

- $\supset R$
- $\wedge L$
- $\wedge R$
- id

Non-invertible rules:

- $\supset L$

2 Focusing

A *focused sequent calculus* attempts to optimize proof search by forcing it to apply invertible rules eagerly, then, once forced into a non-invertible decision, do as much of it as possible all at once.

2.1 Polarizing connectives

We can call connectives with invertible right rules *negative*. We can call connectives with invertible left rules *positive*.

Note that this makes $\wedge$ and p "ambipolar." We could put them in both categories, but for the sake of simpler presentation we will assume $\wedge$ to be positive. However, the choice of polarity for atoms p is interesting, and we will permit atoms of either polarity, annotated as such. Propositional variables A, B without a $+$ or $-$ will stand in for either polarity.

$$\begin{aligned} A^+, B^+ &::= p^+ \mid A \wedge B \\ A^-, B^- &::= p^- \mid A \supset B \end{aligned}$$

2.2 Left focus

When applying a non-invertible rule, we want to *maintain focus* on it for as long as possible in order to reach potential failure and backtrack quickly if needed.

We can choose to enter focus on the left for any negative proposition, as long as we don't already have something in focus (on either the left or the right):

$$\frac{\Gamma \mid [A^-] \Rightarrow C}{\Gamma, A^- \Rightarrow C}\ \mathsf{focus}\text{-}L$$

Then, maintain focus via the left rule for each negative connective:

$$\frac{\Gamma, A \supset B \Rightarrow [A] \quad \Gamma, A \supset B \mid [B] \Rightarrow C}{\Gamma \mid [A \supset B] \Rightarrow C} \supset L^{\circ}$$

Once we hit a positive proposition on the left, we can leave focus:

$$\frac{\Gamma, A^{+} \Rightarrow C}{\Gamma \mid [A^{+}] \Rightarrow C} \ \textsf{blur-}L$$

What about negative atoms? If they appear in focus on the left, the only way to succeed is if they immediately match the goal of the sequent:

$$\frac{}{\Gamma, [p^{-}] \Rightarrow p^{-}} \ \textsf{id}^{-}$$

2.3 Right focus

The focused $\supset L$ rule introduces a "right focus" in one of its subgoals, so we should also say what we can do with that.

For negative propositions, we can only blur.

$$\frac{\Gamma \Rightarrow A^{-}}{\Gamma \Rightarrow [A^{-}]} \ \textsf{blur-}R$$

For positive ones, we must maintain focus on the subformulae.

$$\frac{\Gamma \Rightarrow [A] \quad \Gamma \Rightarrow [B]}{\Gamma \Rightarrow [A \wedge B]} \wedge R^{\circ}$$

Dually to the case for negative atoms, if we have a positive atom in focus on the right, the only way we can succeed is if the same atom is immediately available in the context.

$$\frac{}{\Gamma, p^{+} \Rightarrow [p^{+}]} \ \textsf{id}^{+}$$

If nothing else is in focus, we can also choose to right-focus on a positive formula.

$$\frac{\Gamma \Rightarrow [A^{+}]}{\Gamma \Rightarrow A^{+}} \ \textsf{focus-}R$$

3 Forward and backward chaining, revisited

Consider again the sequent from before, minus the extraneous $s \supset r$ premise:

$$p, p \supset q, q \supset r \Rightarrow r$$

To give a focused proof, we need to choose the polarity for each atom (p, q, r). Let's see what happens in two cases: if we make them all *negative* and if we make them all *positive*.

3.1 Negative atoms induce backward chaining

Goal sequent:

$$p^-, p^- \supset q^-, q^- \supset r^- \Rightarrow r^-$$

If we pick $p^- \supset q^-$ to focus on—the forward-chaining strategy—we will fail, because we will place q^- in focus on the left without it exactly matching the goal:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ p^-, q^- \supset r^-, p^- \supset q^- \Rightarrow [p^-]\end{array} \quad \begin{array}{c}\text{FAIL}\\ p^-, q^- \supset r^-, p^- \supset q^- \mid [q^-] \Rightarrow r^-\end{array}}{p^-, q^- \supset r^- \mid [p^- \supset q^-] \Rightarrow r^-} \supset L^\circ}{p^-, p^- \supset q^-, q^- \supset r^- \Rightarrow r^-}\ \textsf{focus-}L$$

However, if we pick $q^- \supset r^-$ to focus on—the backward-chaining strategy—we can succeed. Applying $\supset L^\circ$ after focus gives us two subgoals:

$$p^-, p^- \supset q^-, q^- \supset r^- \Rightarrow [q^-]$$

and

$$p^-, p^- \supset q^-, q^- \supset r^- \mid [r^-] \Rightarrow r^-$$

The latter succeeds immediately by matching the left focus with the goal:

$$\dfrac{}{p^-, p^- \supset q^-, q^- \supset r^- \mid [r^-] \Rightarrow r^-}\ \textsf{id}^-$$

And the former succeeds by recursively applying the same backward-chaining strategy to get q^-:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p^- \supset q^-, q^- \supset r^- \mid [p^-] \Rightarrow p^-}\ \textsf{id}^-}{p^-, p^- \supset q^-, q^- \supset r^- \Rightarrow p^-}\ \textsf{focus-}L}{p^-, p^- \supset q^-, q^- \supset r^- \Rightarrow [p^-]}\ \textsf{blur-}R \quad \dfrac{}{p^-, p^- \supset q^-, q^- \supset r^- \mid [q^-] \Rightarrow q^-}\ \textsf{id}^-}{\dfrac{p^-, q^- \supset r^- \mid [p^- \supset q^-] \Rightarrow q^-}{\dfrac{p^-, p^- \supset q^-, q^- \supset r^- \Rightarrow q^-}{p^-, p^- \supset q^-, q^- \supset r^- \Rightarrow [q^-]}\ \textsf{blur-}R}\ \textsf{focus-}L} \supset L^\circ}{}$$

3.2 Positive atoms induce forward chaining

Goal sequent:

$$p^+, p^+ \supset q^+, q^+ \supset r^+ \Rightarrow r^+$$

Dually to the negative case, if we choose to focus on $q^+ \supset r^+$, we will fail:

$$\dfrac{\dfrac{\dfrac{\text{FAIL}}{p^+, p^+ \supset q^+, q^+ \supset r^+ \Rightarrow [q^+]} \quad \dfrac{\vdots}{p^+, p^+ \supset q^+, q^+ \supset r^+ \mid [q^+] \Rightarrow r^+}}{p^+, p^+ \supset q^+ \mid [q^+ \supset r^+] \Rightarrow r^+} \supset L^\circ}{p^+, p^+ \supset q^+, q^+ \supset r^+ \Rightarrow r^+} \; \textsf{focus-}L$$

And, again dually to the negative case, if we focus on $p^+ \supset q^+$, we can succeed. The $\supset L^\circ$ rule gives us two subgoals:

$$p^+, p^+ \supset q^+, q^+ \supset r^+ \Rightarrow [p^+]$$

and

$$p^+, p^+ \supset q^+, q^+ \supset r^+ \mid [q^+] \Rightarrow r^+$$

The first succeeds by the positive goal being immediately available in the sequent:

$$\dfrac{}{p^+, p^+ \supset q^+, q^+ \supset r^+ \Rightarrow [p^+]} \; \textsf{id}^+$$

The second succeeds by continuing the forward chaining process:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p^+, p^+ \supset q^+, q^+ \supset r^+, q^+ \Rightarrow [q^+]} \textsf{id}^+ \quad \dfrac{\dfrac{\dfrac{}{p^+, p^+ \supset q^+, q^+ \supset r^+, q^+, r^+ \Rightarrow [r^+]} \textsf{id}^+}{p^+, p^+ \supset q^+, q^+ \supset r^+, q^+, r^+ \Rightarrow r^+} \textsf{focus-}R}{p^+, p^+ \supset q^+, q^+ \supset r^+, q^+, [r^+] \Rightarrow r^+} \textsf{blur-}L}{p^+, p^+ \supset q^+, q^+ \mid [q^+ \supset r^+] \Rightarrow r^+} \supset L^\circ}{p^+, p^+ \supset q^+, q^+ \supset r^+, q^+ \Rightarrow r^+} \textsf{focus-}L}{p^+, p^+ \supset q^+, q^+ \supset r^+ \mid [q^+] \Rightarrow r^+} \textsf{blur-}L$$

4 Credits

All ideas herein are due to prior work: Miller, Nadathur, Pfenning, and Scedrov for the proof-theoretic account of logic programming [4]; Andreolli for the concept of invertible rules and focusing [1]; Chaudhuri, Pfenning, and Price for observing the connection between focusing and forward and backward chaining [2]. My presentational/notational choices are probably closest to those of Liang and Miller's LJF [3] by way of Simmons [5] and Pfenning (unpublished lecture notes).

References

[1] J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of logic and computation*, 2(3):297–347, 1992.

[2] K. Chaudhuri, F. Pfenning, and G. Price. A logical characterization of forward and backward chaining in the inverse method. *Journal of Automated Reasoning*, 40(2):133–177, 2008.

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[4] D. Miller, G. Nadathur, F. Pfenning, and A. Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied logic*, 51(1-2):125–157, 1991.

[5] R. J. Simmons. Structural focalization. *ACM Transactions on Computational Logic (TOCL)*, 15(3):1–33, 2014.