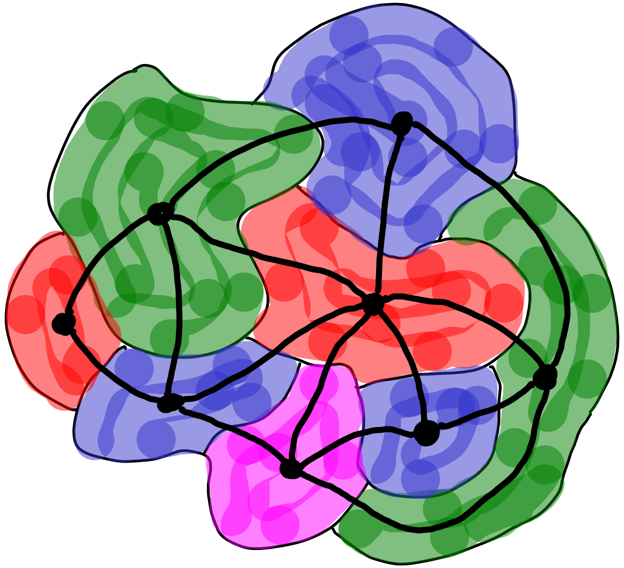
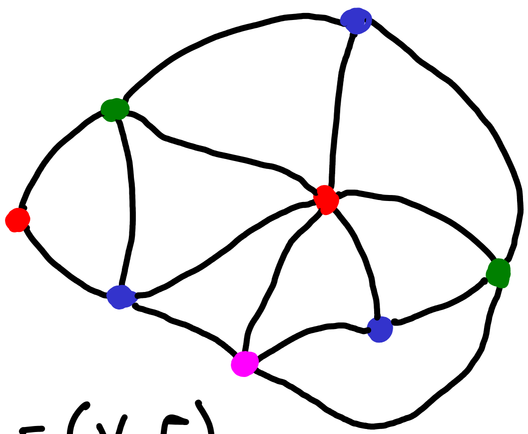
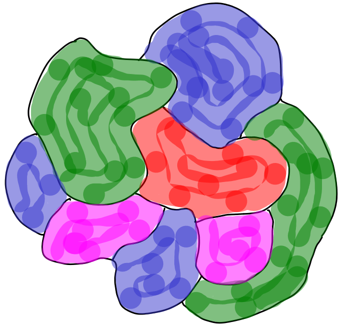
MAP COLORING

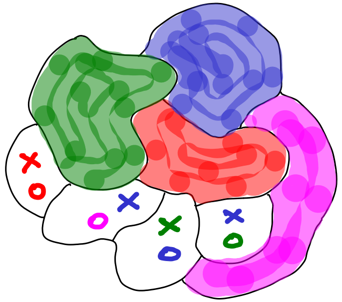


$\Rightarrow$



$G=(V,E)$





etc

COLORING G → no adjacent vertices get same color

G is k-colorable if we can use $\leq k$ colors

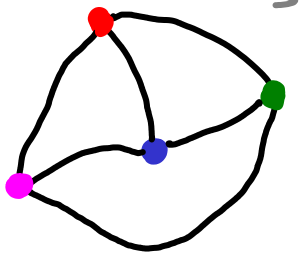
$\chi(G)$: min # colors we can use to color G

chromatic number

χρῶμα = color

Our map is 4-colorable $\Big\}\ \chi \leq 4$

...but not 3-colorable

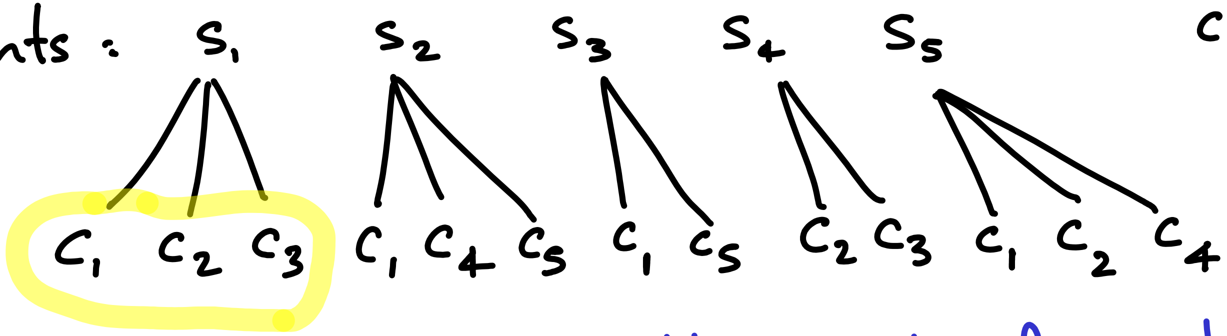


subgraph K_4

so $\chi \geq 4$

EXAM SCHEDULING

students :

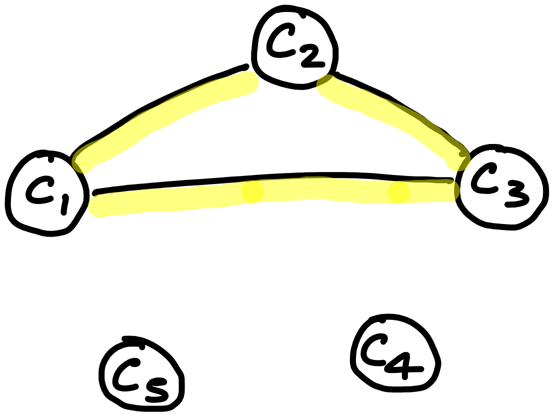


classes c_1 c_2 c_3 c_4 c_5

Can't schedule exam simultaneously for classes taken by s_i

Want to minimize exam slots.

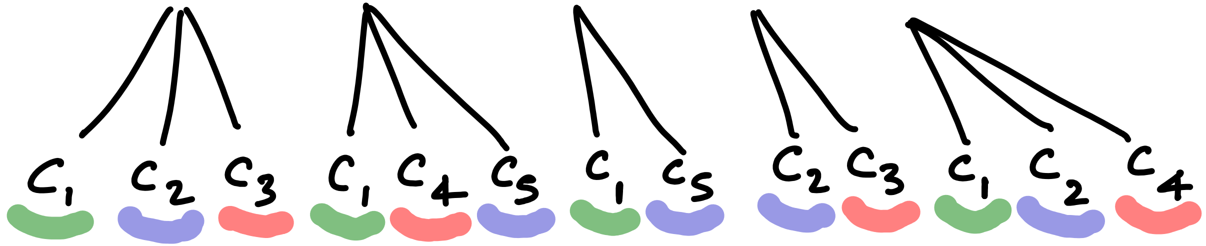
Make G: V = classes E = conflicts



EXAM SCHEDULING

students : s_1 s_2 s_3 s_4 s_5

classes c_1 c_2 c_3 c_4 c_5

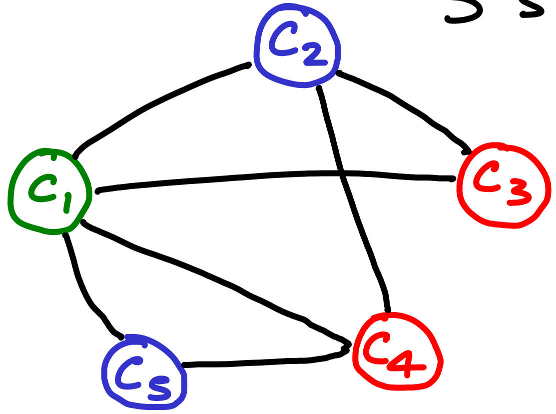


Can't schedule exam simultaneously for classes taken by s_i

Want to minimize exam slots.

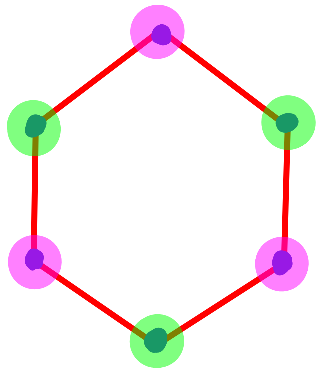
Make G : V = classes E = conflicts

3 slots



Colors = slots (minimize colors)

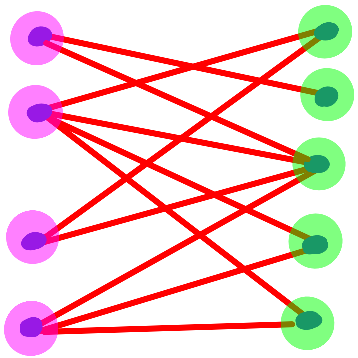
If no edge has same color at endpoints,
then no 2 classes are in same slot



What is χ for cycles?

$\chi = 2$ if V even
$= 3$ if V odd

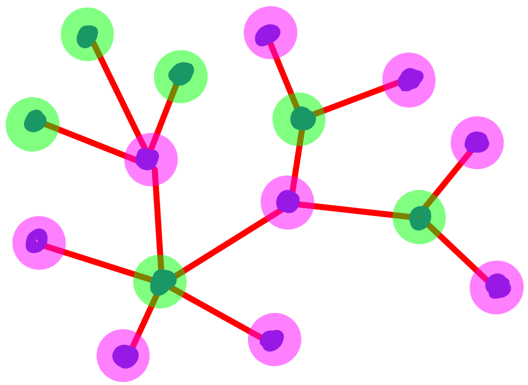
Claim:
G is bipartite
$\Longleftrightarrow$
G contains no odd cycle



For bipartite graphs?

$\chi = 2$

In fact if $\chi(G) = 2$
then G is bipartite
by definition



For trees?

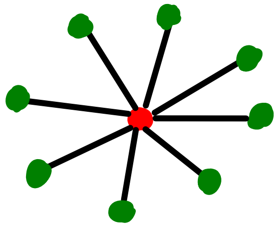
Remove a leaf, v.
2-color the rest.
Color v opposite of p(v)
$\chi = 2$

(trees are bipartite)

If G has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ (K_n)

What can χ be if max degree of $G = \Delta$?

$K_n \rightarrow \Delta = n-1 \;:\; \chi = \Delta + 1$

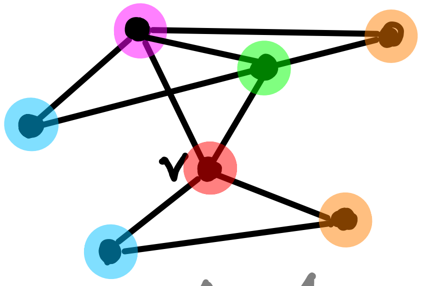


$\Delta = n-1 \;:\; \chi = 2$

Can we have $\chi \gg \Delta$? (need $n \gg \Delta$)

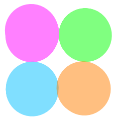
Claim $\chi \leq \Delta + 1$

- Incrementally "add" vertices.
- When adding vertex v, look at all neighboring colors.
- Always have ≥ 1 color available.



$\Delta = 4$

- Remove any vertex v.
- Color $G - v$ by induction.
- Re-insert v.
- v has $\leq \Delta$ neighbors.
- Use color $\Delta + 1$ for v.



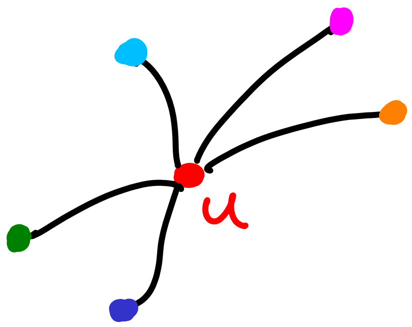
COLORING PLANAR GRAPHS (like map duals)

Claim: $\chi \leq 6$ ... trivial if $V \leq 6$

We know planar graphs have a vertex w/ degree ≤ 5

Given planar G s.t. $V>6$ & $u \in G$, $d(u) \leq 5$: look at $G-u$

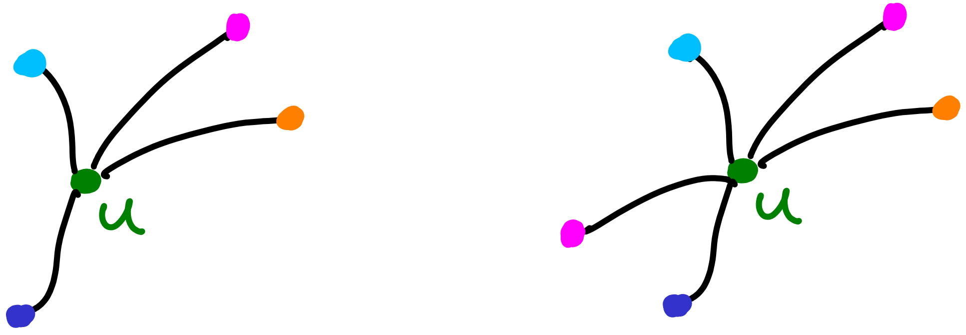
Assume by induction that $G-u$ is 6-colorable



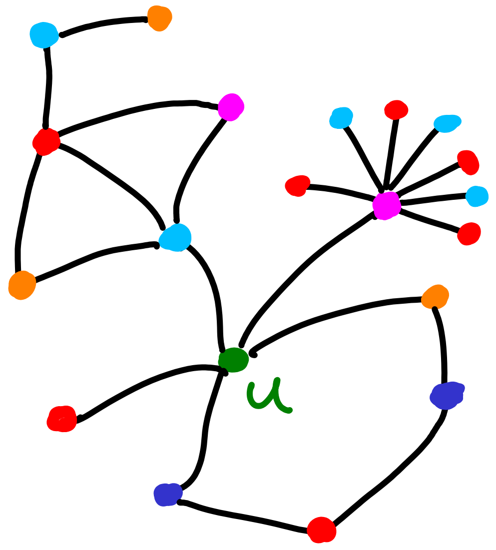
re-insert u : give it a color not used by neighbors $\square$

Claim: $\chi \leq 5$ ... trivial if $V \leq 5$

Use induction & $d(u) \leq 5$ again

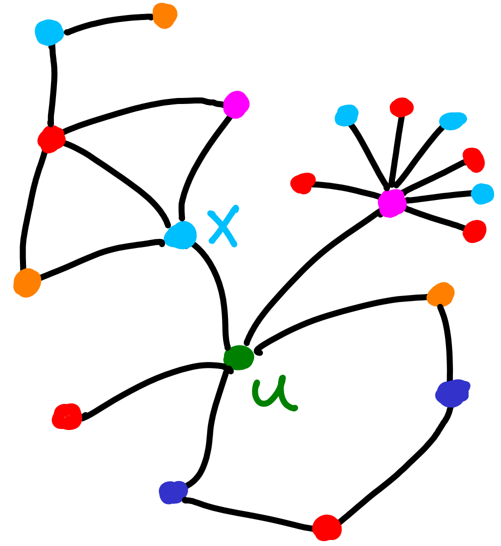


Also trivial if neighbors use < 5 colors



Consider any embedding of G

We need a neighbor of u to change color

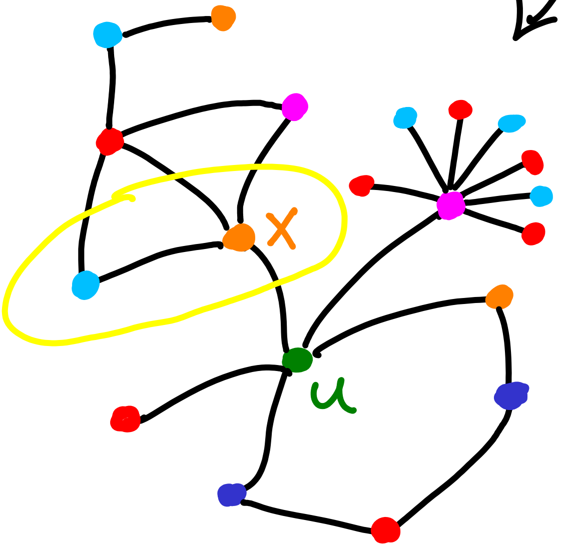


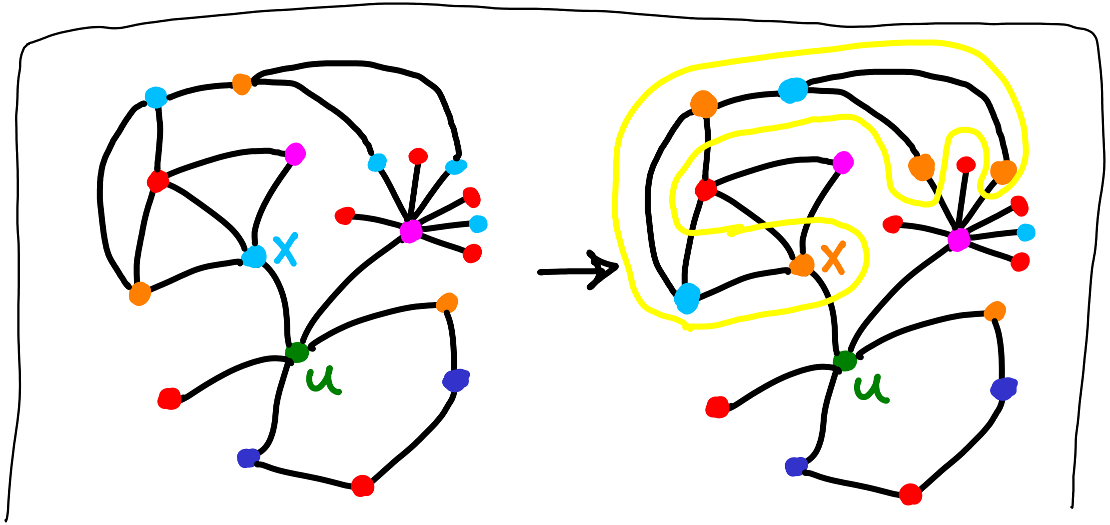
Try to change x from to

[specifically skipping 2 over in adj(u)]

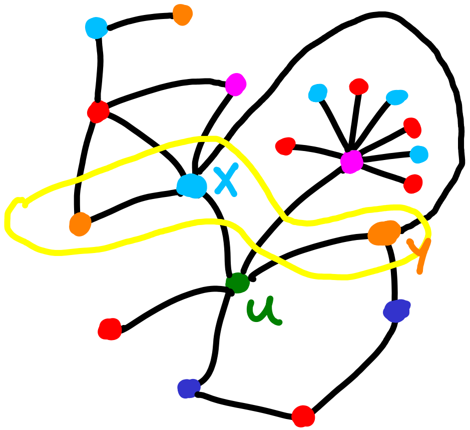
↳ This works if x has no neighbors

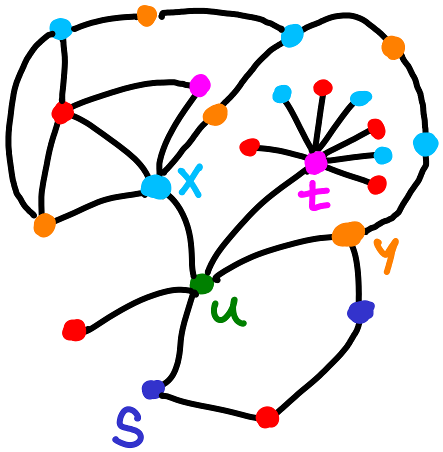
↳ else, swap colors on the connected component of the subgraph of G that contains only colors and x.





ONE PROBLEM





The only bad case involves a path from x to y
that alternates x●●●●...y

Together with u the path forms a cycle
surrounding the ● neighbor of u.

Restart the entire procedure using s & t instead of x & y.

The only way to fail is if there is a path s●●●●t
but this would have to cross x●●●●...y

(Impossible: This is a plane drawing

□

Planar graphs:

6-coloring: ~ trivial

5-coloring: short elegant proof

4-coloring:
- unsolved from ≤ 1850 until 1977
- proof involved ~2000 cases solved by computer

3-coloring:
- clearly not always possible
- if triangle-free then 3-colorable

(in fact if ≤ 3 triangles)