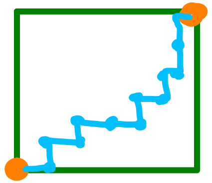
COUNTING in this course so far

- number of subsets of a set of n elements: 2^n
- Cartesian product
- recursive sums
- levels, leaves & nodes in binary trees
- first n odd positive integers sum to n^2
- $\sum_{i=0}^{n} 3^i < 3^{n+1}$ / $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ / $\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$ / $\sum_{x=1}^{n} x^9 = \Theta(n^{10})$
-  number of paths $= \frac{(n+m)!}{n!\,m!}$

SUMMATIONS - not an introduction

recommended: become familiar with basic notation & properties

$$\sum_{i=1}^{n} c \cdot f(i) = c \cdot \sum_{i=1}^{n} f(i)$$

$$\sum_{i=1}^{n} \left(f(i) + g(i)\right) = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$$

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{x} f(i) + \sum_{i=x+1}^{n} f(i)$$

$$\sum_{i=x}^{x-1} f(i) = 0$$

$$\sum_{i=1}^{n} \sum_{j=1}^{i} j$$

nested sums

$$\sum_{i=1}^{n} f(i) = \sum_{i=1+x}^{n+x} f(i-x)$$

etc

This set of notes focuses on certain sums

encountered often in CS

Geometric sum

Arithmetic sum

Harmonic sum

Telescoping sum

Arithmetic sum $\displaystyle\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$ (see proof in induction notes)

proof by Gauss:

$$\sum_{x=1}^{n} x = \sum_{x=1}^{n} (n-x+1)$$

$$2\cdot\sum_{x=1}^{n} x = \sum_{x=1}^{n} x + \sum_{x=1}^{n} (n-x+1)$$

$$1+2+3+\dots+(n-1)+n$$

$$\underbrace{n+(n-1)+\dots+3+2+1}_{(n+1)+(n+1)+(n+1)+\cdots}$$

$$= \sum_{x=1}^{n} \left[x+(n-x+1)\right]$$

$$= \sum_{x=1}^{n} (n+1)$$

$$= n(n+1)$$

Telescoping sum

$$\sum_{i=1}^{n} (F_i - F_{i-1}) = F_n - F_0$$

given $n+1$ values, F_i $(0 \le i \le n)$

e.g., $\overbrace{2i^3 + i \quad \text{or} \quad 4^i}$

$i=n:$ $\quad F_n - \cancel{F_{n-1}}$

$i=n-1:$ $\quad + \cancel{F_{n-1}} - \cancel{F_{n-2}}$

$i=n-2:$ $\quad + \cancel{F_{n-2}} - \cancel{F_{n-3}}$

$\vdots \qquad \vdots$

$i=2:$ $\quad + \cancel{F_2} - \cancel{F_1}$

$i=1:$ $\quad + \cancel{F_1} - F_0$

$$\sum_{i=1}^{n} (F_i - F_{i-1}) = \sum_{i=1}^{n} F_i - \sum_{i=1}^{n} F_{i-1}$$

$$= \sum_{i=1}^{n} F_i - \sum_{i=0}^{n-1} F_i$$

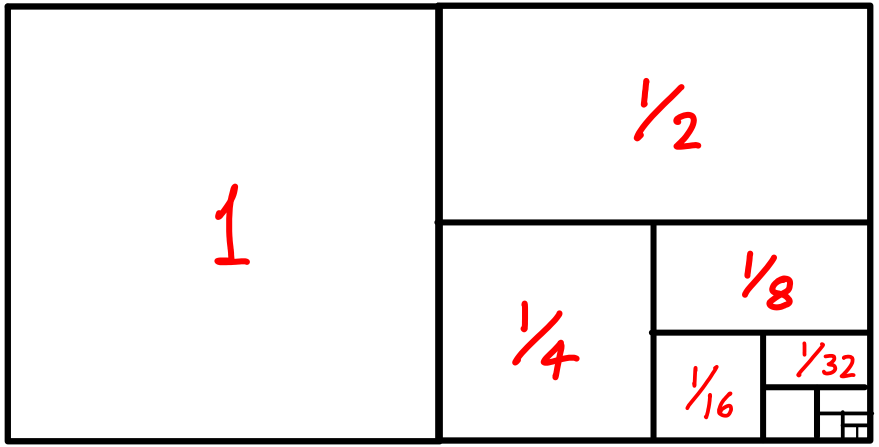
$$= F_n + \sum_{i=1}^{n-1} F_i - \sum_{i=1}^{n-1} F_i - F_0$$

$$= F_n - F_0$$

Geometric sum $\sum_{i=0}^{n} x^i$ $\quad x \in \mathbb{R}^+$ & $x \neq 1$

for $x=1$, $\sum_{i=0}^{n} x^i = n+1$ (not geometric)

$x = \frac{1}{2}$: $\quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$



Geometric sum $\sum_{i=0}^{n} x^i$ $\quad x \in \mathbb{R}^+$ & $x \neq 1$

for $x=1$, $\sum_{i=0}^{n} x^i = n+1$ (not geometric)

$x=\frac{1}{2}$: $\quad 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots < 2$

$x=2$: $\quad 1+2+4+8+16+\cdots = 2^{n+1}-1$

$$x \cdot \sum_{i=0}^{n} x^i = \sum_{i=0}^{n} x^{i+1} = \sum_{i=1}^{n+1} x^i$$

$$x \cdot \sum_{i=0}^{n} x^i - \sum_{i=0}^{n} x^i = (x-1) \cdot \sum_{i=0}^{n} x^i$$

$$= \sum_{i=1}^{n+1} x^i - \sum_{i=0}^{n} x^i = x^{n+1} - x^0$$

telescoping

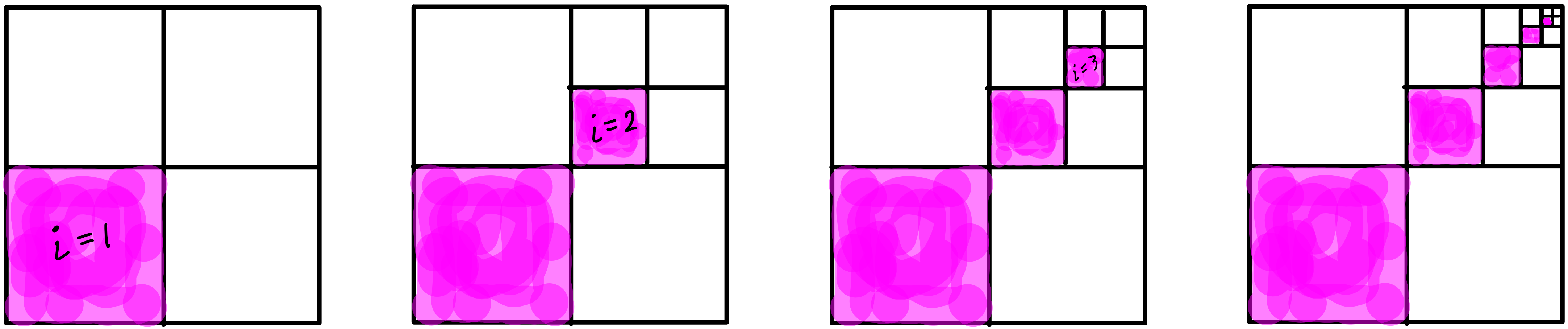
often used when $x<1$

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1} = \frac{1-x^{n+1}}{1-x}$$

$\Sigma =$	$x>1$	$x<1$	
	$\Theta(x^n)$	$\Theta(1)$	$\rightarrow \Theta$(largest term)

if $n \to \infty$, $x<1$: $\quad \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

$x < 1$ $\qquad \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ $\qquad \sum_{i=1}^{\infty} x^i = \frac{1}{1-x} - 1 = \frac{x}{1-x}$



$$i: \quad 1 \quad 2 \quad 3 \quad 4$$

$$\sum_{i=1}^{\infty}\left(\frac{1}{4}\right)^i = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}$$

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$ by induction

hypothesis : $$\sum_{i=0}^{n-1} x^i = \frac{x^n-1}{x-1}$$

$$\sum_{i=0}^{n} x^i = x^n + \sum_{i=0}^{n-1} x^i$$

$$= x^n + \frac{x^n-1}{x-1}$$

$$= \frac{(x-1)x^n}{x-1} + \frac{x^n-1}{x-1}$$

$$= \frac{x^{n+1}-x^n}{x-1} + \frac{x^n-1}{x-1} = \frac{x^{n+1}-1}{x-1}$$

base case, $n=0$

$$\sum_{i=0}^{0} x^i = 1$$

$$\frac{x^{0+1}-1}{x-1} = \frac{x-1}{x-1} = 1$$

$\square$

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

if $n \to \infty$ $\quad \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

$$0.\overline{99} = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$$

$$= 0.9 \cdot \left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$

$$= 0.9 \cdot \sum_{i=0}^{\infty} \left(\frac{1}{10}\right)^i$$

$$= 0.9 \cdot \frac{1}{1 - 1/10}$$

$$= 1$$

other formats you might see

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x-1} = \frac{1 - x^{n+1}}{1-x}$$

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1} = \frac{1 - x^n}{1-x} = \sum_{i=1}^{n} x^{i-1}$$

$$\sum_{i=0}^{n} a \cdot x^i = a\frac{x^{n+1} - 1}{x-1} = a\frac{1 - x^{n+1}}{1-x}$$

for $x < 1$

$$\sum_{i=0}^{\infty} i \cdot x^i = \frac{x}{(1-x)^2}$$

by taking derivative

Harmonic sum $\sum_{i=1}^{n} \frac{1}{i}$ = n-th harmonic number

No known closed form. Only approximations.

Can be shown:

$$\underbrace{\int_1^n \frac{1}{x}\,dx}_{\text{close to } \sum_{i=1}^{n} \frac{1}{i}} = \ln x \Big|_1^n = \ln n$$

$$\ln n + \frac{1}{n} \leq \sum_{i=1}^{n} \frac{1}{i} \leq \ln n + 1$$

- even better approximations exist

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interesting to look up: max overhang problem