Concepts used in this document

- ceiling function (round up) e.g., $\lceil 1.5 \rceil = 2$
- Set, Subset
- # of subsets that can be formed from a set of size n $= 2^n$
- $\sum_{i=0}^{n} 2^i = 2^n - 1$
- Contrapositive, proof by contradiction

THE PIGEONHOLE PRINCIPLE

If n holes are occupied by $n+1$ pigeons,
then at least one hole is occupied by at least two pigeons.

THE PIGEONHOLE PRINCIPLE

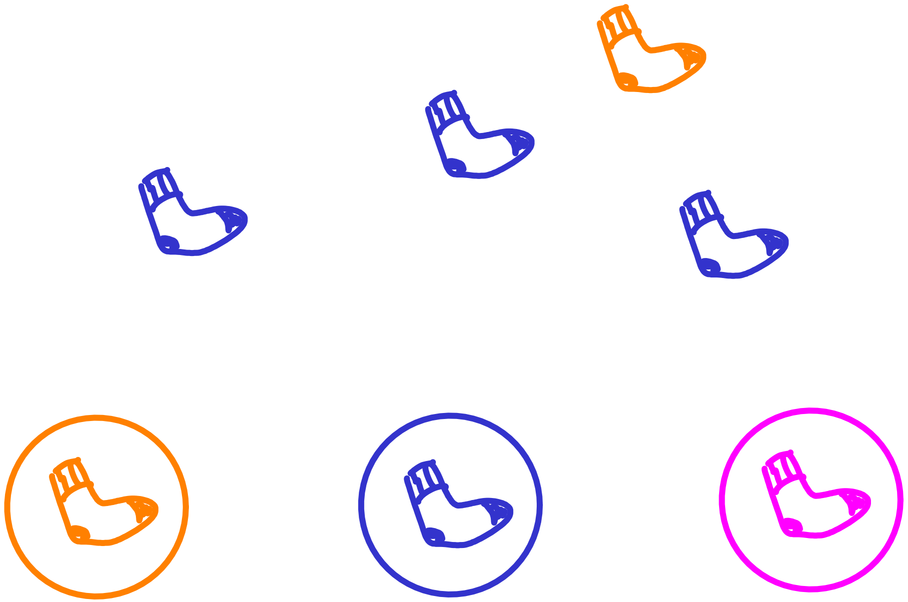
If n holes are occupied by p pigeons, $(p > n)$, then one hole is occupied by at least $\frac{p}{n}$ pigeons.

$$\left.\begin{matrix} n=6 \\ p=9 \end{matrix}\right\} \text{ at least } 1.5 \rightarrow \text{ at least } 2 = \lceil p/n \rceil$$

THE PIGEONHOLE PRINCIPLE

If you have a pile of socks, of 3 colors,
how many do you need to pick (randomly) to get a matching pair?

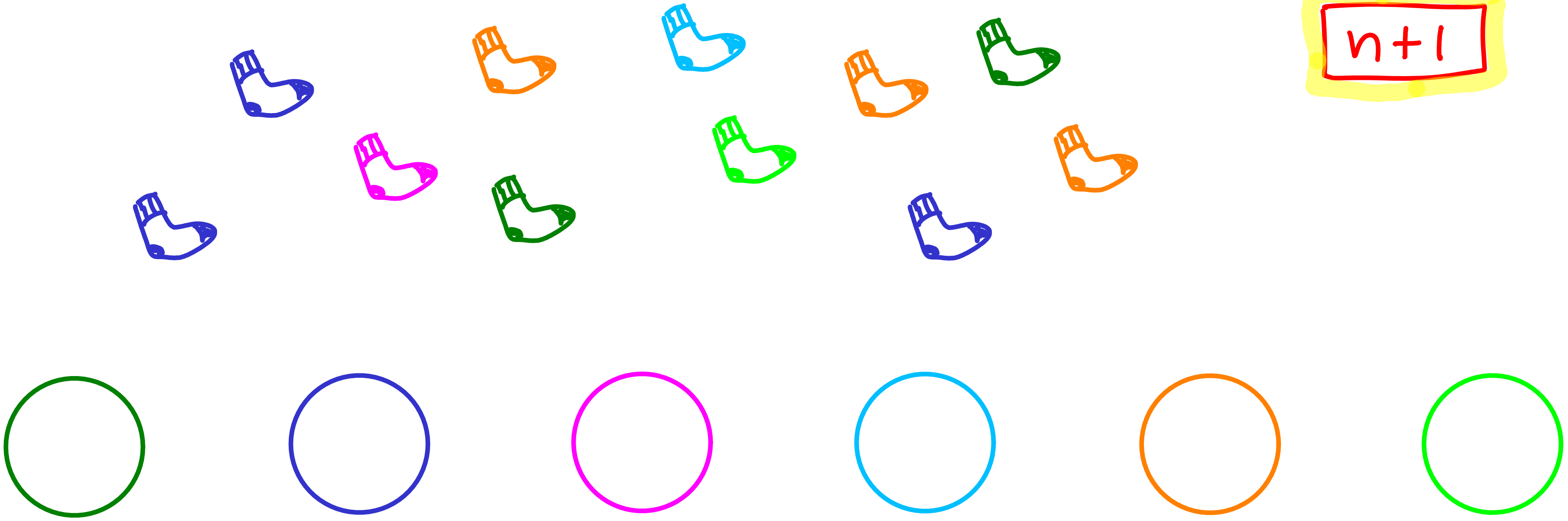
$\boxed{4}$



THE PIGEONHOLE PRINCIPLE

If you have a pile of socks, of n colors, how many do you need to pick (randomly) to get a matching pair?

$\boxed{n+1}$



THE PIGEONHOLE PRINCIPLE

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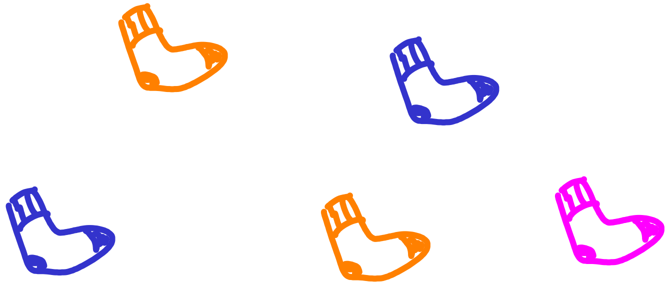


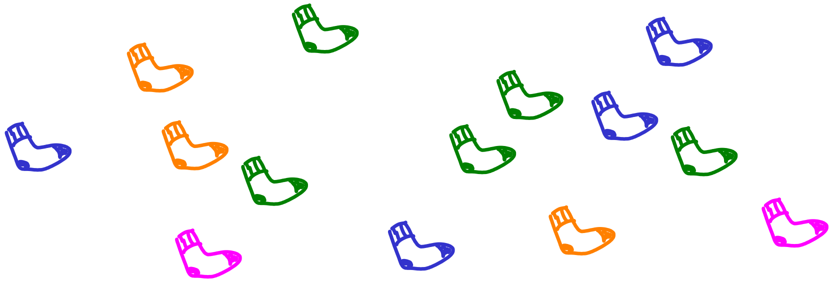
$\boxed{2n+1}$

Worst scenario: pick 2 of each color $= 2n$
before getting a triple.

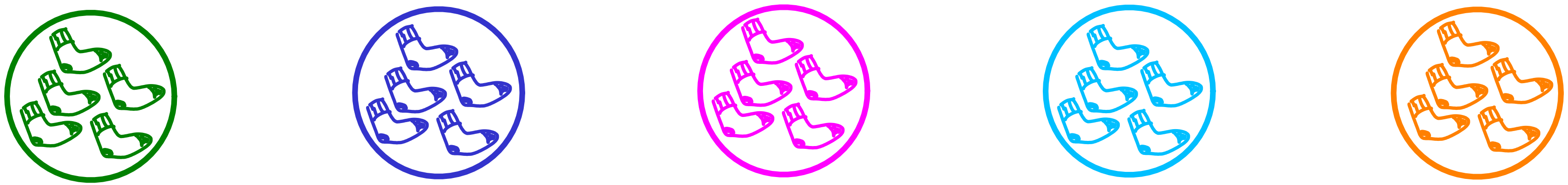
THE PIGEONHOLE PRINCIPLE

If you have a pile of socks, of n colors,
how many do you need to pick (randomly) to get k of one type?

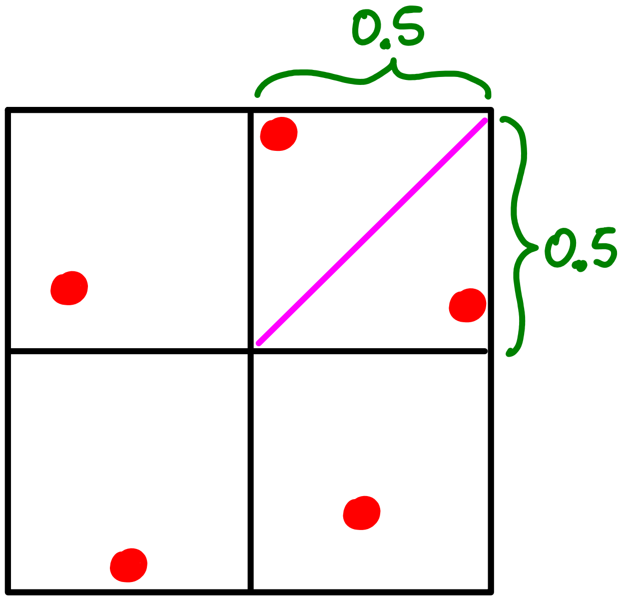
$$\boxed{(k-1)\cdot n+1}$$



Worst scenario:
pick $\underline{k-1}$ of each color...



Prove: for any set of 5 points in a unit square, there are 2 points within distance $\leq \frac{\sqrt{2}}{2}$



There are 4 quadrants (holes)

By pigeonhole, ≥ 2 points are in one quadrant.

points = pigeons

Max distance in quadrant $= \sqrt{(0.5)^2 + (0.5)^2} = \frac{\sqrt{2}}{2}$ $\square$

Works for other shapes & dimensions too.

Prove: if n teams play each other once (aka round robin),
and every team wins at least once, [no ties allowed]
then 2 teams will have the same number of wins.

How many wins could a team have? $\{1, 2, \ldots, n-1\}$

n teams = pigeons

$n-1$ possible wins = holes

$\square$

Definition: if A is a friend of B then B is a friend of A.

Prove: at any party with n people,
two of them have the same number of friends present.

How many friends could a person have at the party? $\{0, 1, 2, \ldots, n-1\}$

But if someone has no friends, then nobody has $n-1$ friends.

So the set is $\{0, 1, 2, \ldots, n-2\}$ OR $\{1, 2, \ldots, n-1\}$

By pigeonhole, $\left[\begin{array}{l} \text{people} = \text{pigeons} \\ \text{valid \#friends} = \text{holes} \end{array}\right]$ either case works

$\square$

$S = \{1, 2, 3, \dots, 100\}$

Prove: if you are given any 51 numbers from S, you can find a pair that sums to 101.

Make 50 buckets: $\{1, 100\}, \{2, 99\}, \dots, \{49, 52\}, \{50, 51\}$

By pigeonhole, your 51 numbers must include 2 in the same bucket.

$\square$

Let L be a list of 32 8-digit decimal numbers.

31432561
44519287
72911673
⋮

Prove: there are 2 subsets of L that have the same sum.

Range of possible sums? → $0 \ldots \sum_L$ → $0 \ldots 32 \cdot 10^8$

$< 3{,}200{,}000{,}000$ "holes"

How many subsets can we make? → $\underline{2}^{32} = 4{,}294{,}967{,}296$ "pigeons"

(all combinations of in/out)

By pigeonhole, 2 subsets must have the same sum. □

Note: if 2 subsets have common numbers we can remove them & get a solution with 2 disjoint subsets.

Let L be a list of 32 8-digit decimal numbers.

31432561
44519287
72911673
⋮

Proved: there are 2 subsets of L that have the same sum.

For 25-digit numbers, you only need $|L| \geq 90$
(see MCS, ch.15)

It is difficult to actually find a solution efficiently
but it was easy to show that a solution exists.

This type of proof is called "non-constructive"

This problem is related to applications from shipping/packaging to crypto.

Prove: for every n there are integers $1 \le a, b \le 11$ s.t. $a \ne b$, and $10 \mid \underbrace{n^a - n^b}_{\text{divisible by 10}}$

example 1: $n=3$. Pick $a=5$, $b=1$. $3^5 - 3^1 = 240$

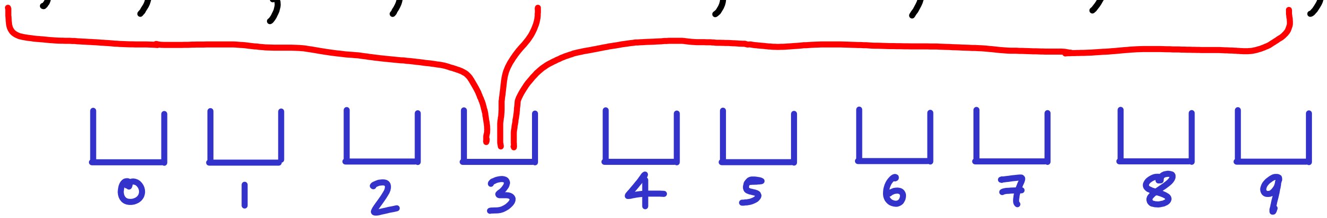
example 2: $n=4$. Pick $a=5$, $b=3$. $4^5 - 4^3 = 960$

example 3: $n=17$. Pick $a=6$, $b=2$. $17^6 - 17^2 = 24{,}137{,}280$

We want $n^a - n^b$ to end with a 0. $\rightarrow$ n^a & n^b end with same digit.

Compute $n^0, n^1, n^2, \ldots, n^{10}$ and place each in a bucket by last digit.

$$\text{e.g. } n=3: \quad \overset{0}{1},\ \overset{1}{3},\ \overset{2}{9},\ \overset{3}{27},\ \overset{4}{81},\ \overset{5}{243}\ \overset{6}{729},\ \overset{7}{2187},\ \overset{8}{6561},\ \overset{9}{19683},\ \overset{10}{59049}$$



Proved by pigeonhole. 11 powers (pigeons), 10 buckets (holes) □

FILE ZIPPING

Objective: reduce storage (#bits) for any given file

$$\text{files}\ \begin{matrix} A \\ B \end{matrix} \to \begin{matrix}\text{zip via any single} \\ \text{zipping algorithm}\end{matrix} \to \underbrace{\begin{matrix} A.\text{zip} \\ B.\text{zip} \end{matrix} \to \begin{matrix}\text{unzip} \\ \text{when needed}\end{matrix} \to \begin{matrix} A \\ B \end{matrix}}_{\text{if } A.\text{zip} = B.\text{zip},\ \text{can't unzip reliably}}$$

Claim: Whatever zip algorithm you choose (with reliable unzip), there is some n-bit file X for which X.zip uses $\geq n$ bits

Contrapositive: if all .zip files use $< n$ bits, unzip won't work reliably.

Look at all n-bit files. How many? 2^n (pigeons)

How many zip files with $< n$ bits? $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ (holes)

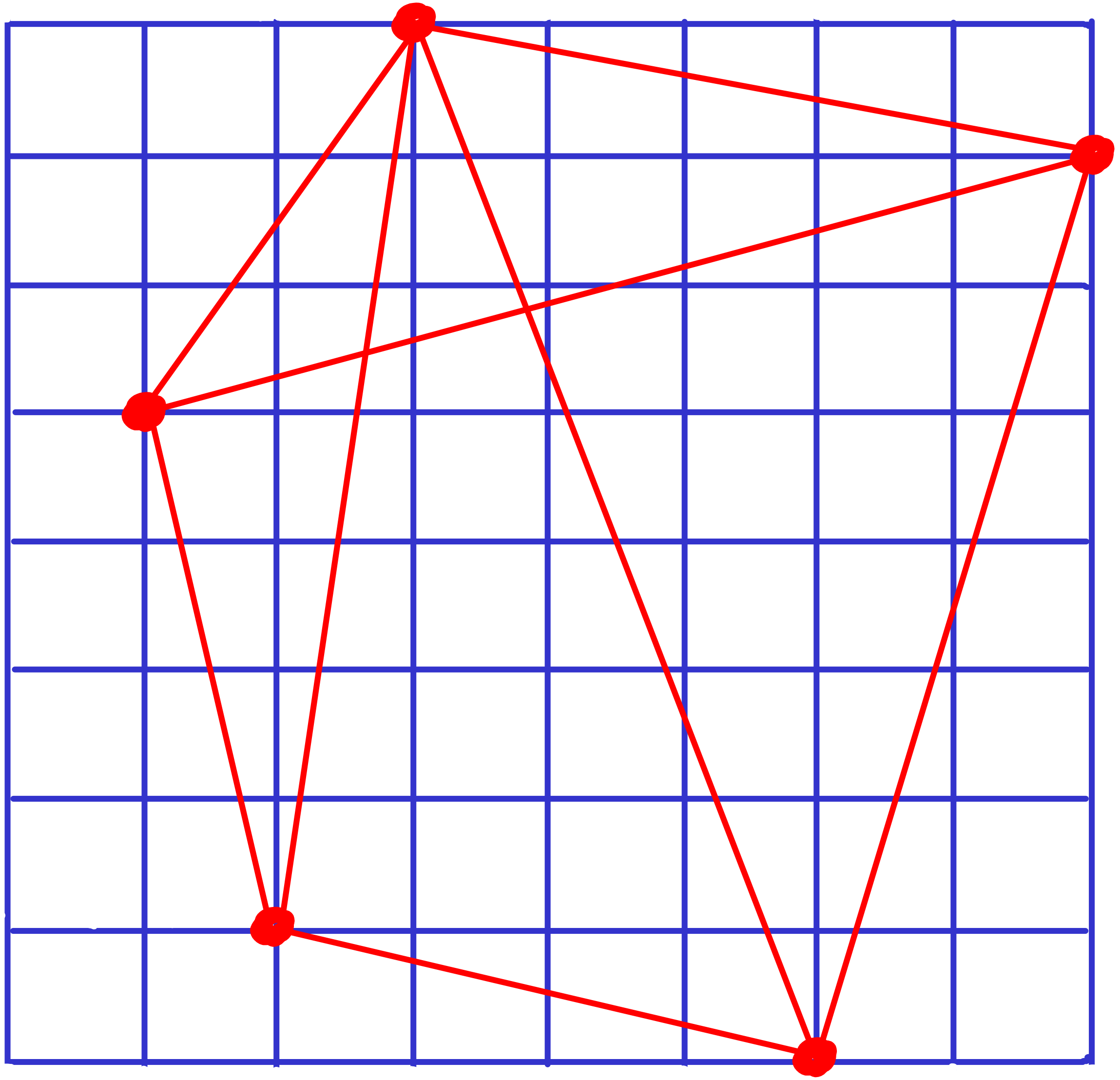
!

If every n-bit file zips to a $<$n-bit file, $\exists\, A, B$ s.t. $A.\text{zip} = B.\text{zip}$ $\square$

Consider 5 points on a grid.
(size & dimensions don't matter)

Prove: $\exists$ 2 points such that their midpoint is on the grid.

Example:



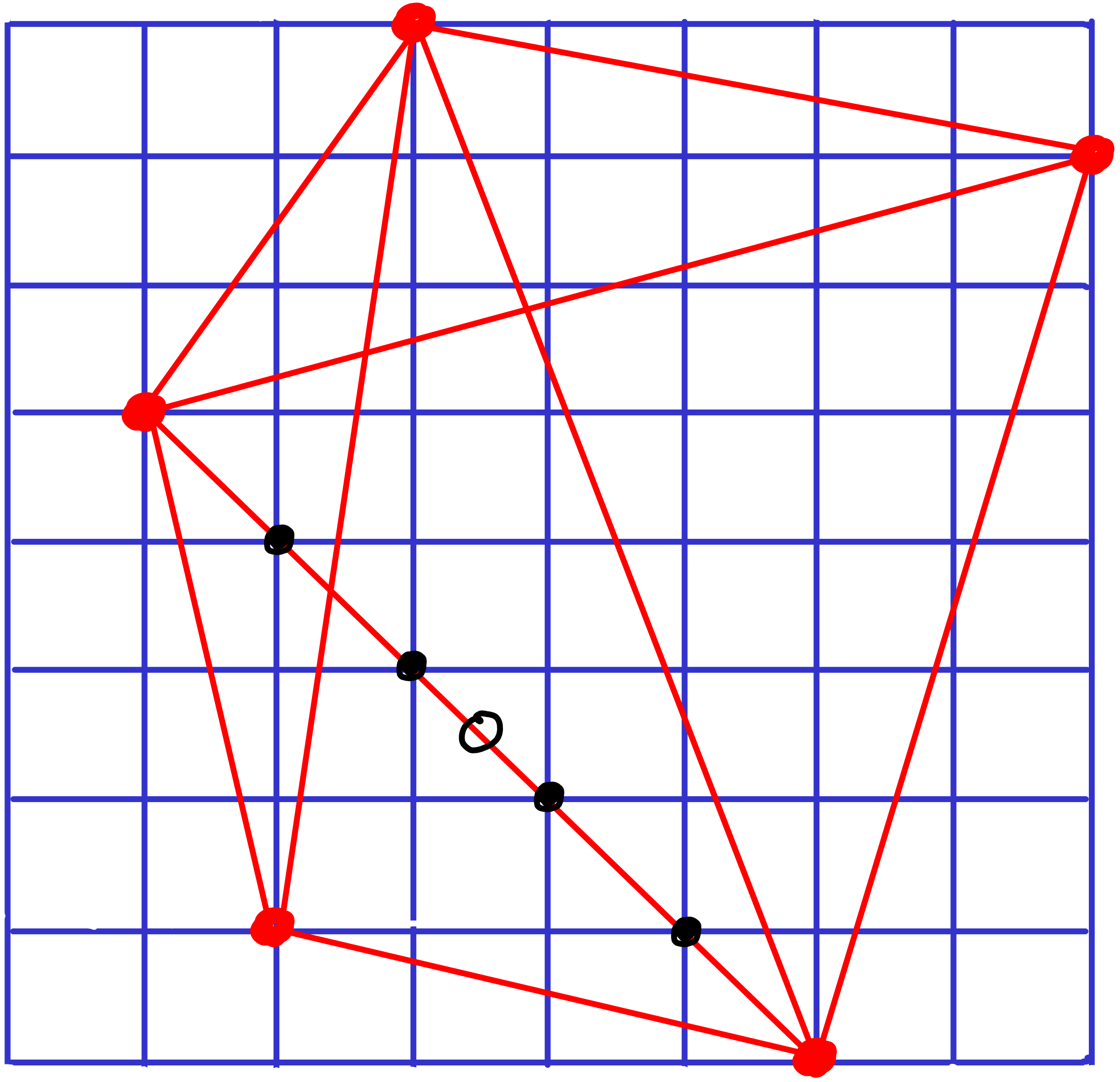
✗ segments don't pass through grid point

Consider 5 points on a grid.
(size & dimensions don't matter)

Prove: $\exists$ 2 points such that their midpoint is on the grid.

Example:

$\times$ segment does pass through grid points but midpoint isn't on grid.



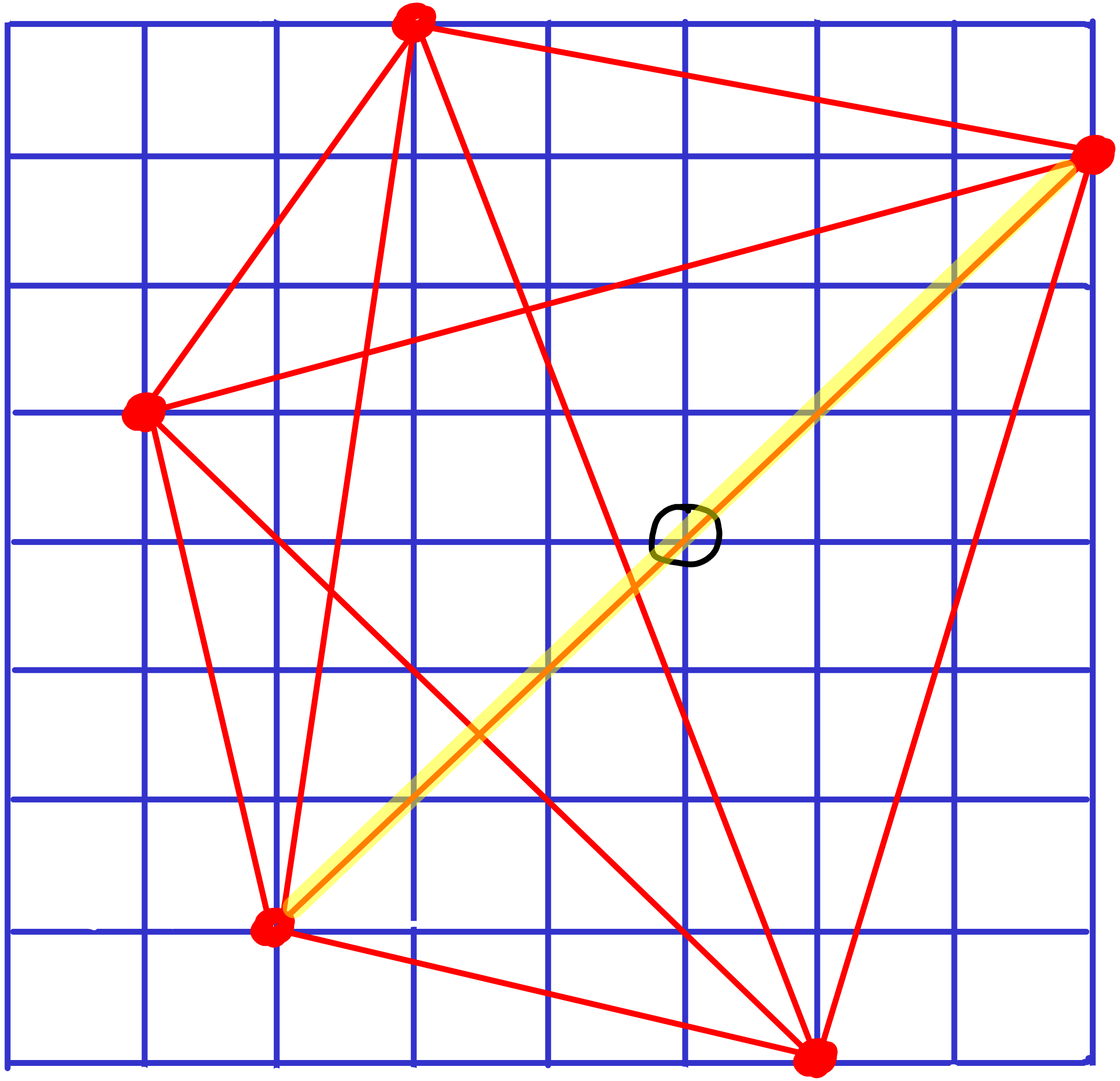
Consider 5 points on a grid.
(size & dimensions don't matter)

Prove: $\exists$ 2 points such that their midpoint is on the grid.

Example:

✓ success

(try to shift points & avoid the midpoint claim)



Prove: among any 5 grid points, ≥ 2 have midpoint on grid.

4 grid position types for (x,y):

odd v Even: (O,E) (E,O) (O,O) (E,E)

Pigeonhole: 2 points A,B → same type.

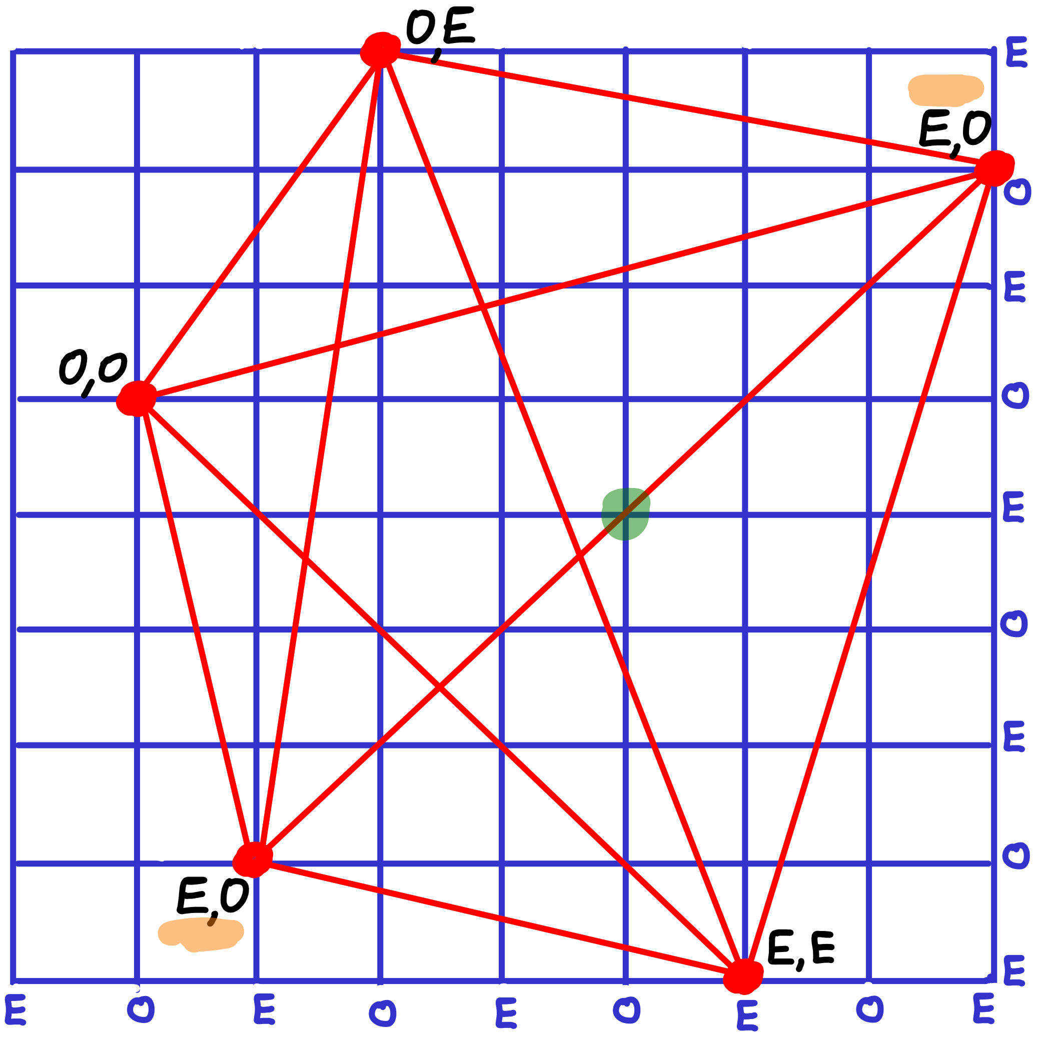
For each coordinate $c = \{x, y\}$

$$\text{midpoint}(A,B)_c = \frac{1}{2}(A_c + B_c)$$

If same type, $(A_c + B_c) \to$ even $\to 2 \cdot k$

(e.g., odd + odd = even) (k = integer)

So midpoint $= \frac{1}{2} 2k =$ integer $\square$



Given a sequence of distinct numbers, a subsequence of size k is k-monotone if it is either <u>increasing</u> or <u>decreasing</u>.

8-monotone

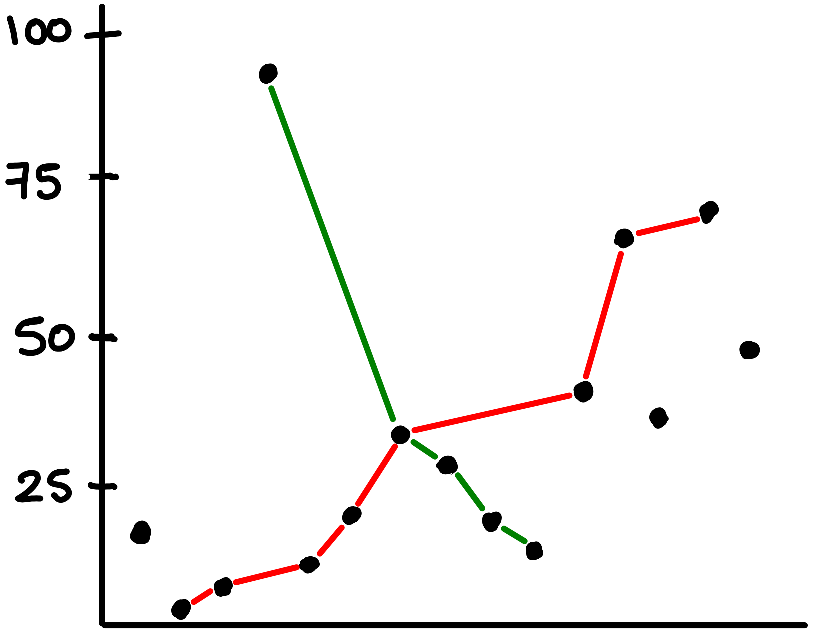
17 3 6 92 8 22 31 27 18 13 45 68 33 72 49

5-monotone

(6,22) (8,27)

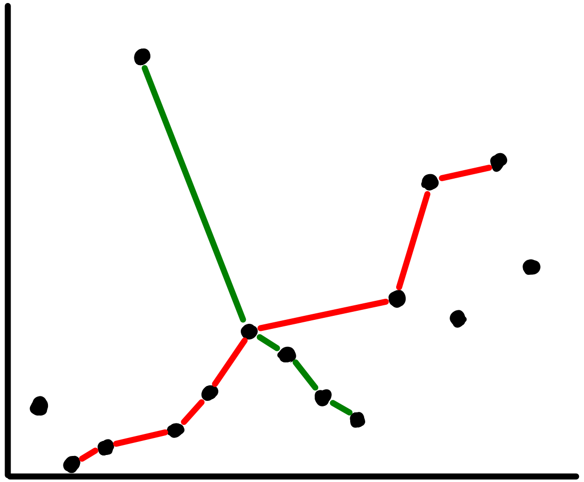
Geometric view:

Create 2D point set



How large must a sequence be, to guarantee a "large" monotone subsequence?

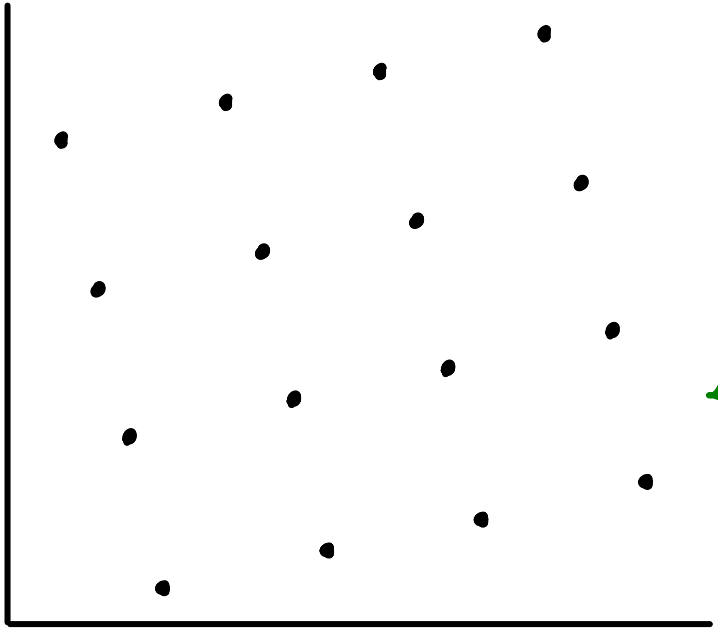
Claim: A $(n+1)$-monotone subsequence exists in every sequence of size n^2+1.



Notice, we can't do better:

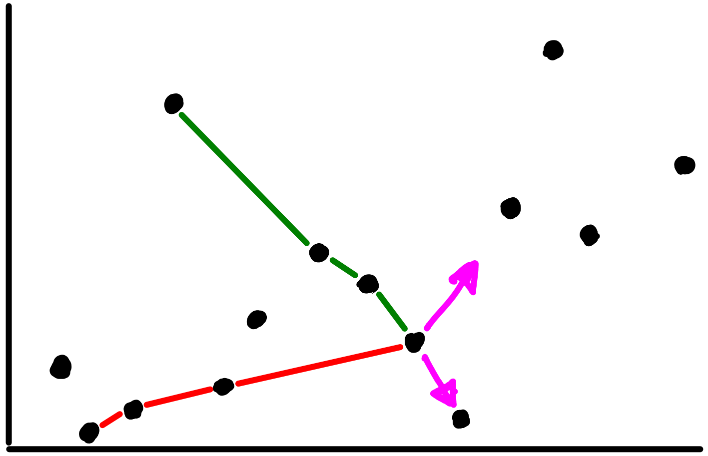
$n=4 \rightarrow n^2 = 16$ pts:

can't find $(n+1)$-monotone $= 5$-monotone subsequence *



With 1 more point, we can. *

Prove: A $(n+1)$-monotone subsequence exists in every sequence of size n^2+1.



Define u_i, d_i : sizes of longest upward & downward chains ending at position i.

Assuming no $(n+1)$-monotone chain, for all i : $u_i, d_i \leq n$ $\Rightarrow \leq n^2$ holes

By pigeonhole, 2 of the $\underbrace{n^2+1 \text{ points}}_{\text{pigeons}}$ have equal (u,d).

But...

position $j = i+k$ will extend u_i or d_i $\rightarrow$ either $u_j > u_i$ or $d_j > d_i$

↳ So for all i,j: $(u_i, d_i) \neq (u_j, d_j)$ Contradiction. $\square$