It is assumed that you know:

$\exists$, $\mathbb{Z}$, $\mathbb{N}$

even + even = even

odd + odd = even

even + odd = odd

iff

$$\sum_{x=1}^{0} x = 0$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

$$0! = 1$$

not critical

n-choose-k: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

PROOF by INDUCTION

Suppose you have n dominoes in a row.

FACT: If the k-th domino falls to the right, then so does the $(k+1)$-st

Claim: if we tip the 1st domino, then eventually the n-th falls

Claim: if we tip the 1st domino, then eventually the n-th falls

FACT: If the k-th domino falls, then so does the (k+1)-st

Apply FACT: if the 1st domino falls, then so does the 2nd.

(again) ... if the 2nd domino falls, then so does the 3rd.

...

...

... if the (n-1)-st domino falls, then so does the n-th.

Claim: if we tip the 1st domino, then eventually the n-th falls

FACT: If the k-th domino falls, then so does the (k+1)-st

Assuming that our claim is true for n-1 ...

"if we tip the 1st domino, then eventually the (n-1)-st falls"

Apply FACT once: if the (n-1)-st domino falls, then so does the n-th.

... we prove original claim.

Claim: if we tip the 1st domino, then eventually the n-th falls

Using FACT,

~~assuming~~ our claim is true for 2, then...

⋮

~~assuming~~ our claim is true for n-2, then...

~~assuming~~ our claim is true for n-1, then...

we prove claim for n

PROOF by INDUCTION

You want to prove something involving n. "n will fall"

- Assume that it's true if you have n-1 instead. "n-1 will fall"
- Prove that this assumption helps solve the actual problem. $n-1 \rightarrow n$
- Don't forget to push the first domino!
  - Prove what you need for a small value $1 \rightarrow 2$

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

- Assume that it's true if you have n-1 : $\sum_{i=0}^{n-1} 3^i < 3^{(n-1)+1}$

- Prove that this assumption helps solve the actual problem.

$$\sum_{i=0}^{n} 3^i = 3^n + \sum_{i=0}^{n-1} 3^i < 3^n + 3^{(n-1)+1} = 3^n + 3^n < 3 \cdot 3^n = 3^{n+1}$$

- Prove what you need for a small value (n=0) → $\sum_{i=0}^{0} 3^i = 3^0 < 3^{0+1}$

PROOF by INDUCTION

You want to prove something involving n

- Assume that it's true if you have n-1 — Inductive hypothesis
- Prove that this assumption helps — Inductive step
- Prove what you need for a small value — Base case

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n = 0$ will be base case

By definition, the assumption is: $\exists a \in \mathbb{Z}$ such that $3a = 4^{n-1} - 1$

$\hookrightarrow \; 4 \cdot 3a = 4 \cdot (4^{n-1} - 1) = 4^n - 4$

$\hookrightarrow \; 3 \cdot 4a + 3 = 4^n - 1$

$\hookrightarrow \; 3\underbrace{(4a+1)}_{\in \mathbb{Z}} = 4^n - 1 \rightarrow$ By definition, $3 \mid 4^n - 1$ ✓

Base case: $4^0 - 1 = 0$, so $3 \mid 0$ is true ✓

$\square$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $n=5$ will be base case

$$32 = 2^5 > 5^2 = 25$$

$$2^{n-1} > n^2 - 2n + 1$$

$$2 \cdot 2^{n-1} > 2n^2 - 4n + 2$$

$$2^n > n^2 + (n^2 - 4n + 2)$$

$$> n^2 + (n^2 - 4n - 5)$$

$$= n^2 + \underbrace{(n-5)\cdot(n+1)}_{>0}$$

$$> n^2$$

$\square$

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

Prove: for all n, $F_n \leq 1.7^n$

Base cases: $n=1$ & $n=2$, trivially true

Hypothesis: for $n \geq 2$ assume $F_{n-1} \leq 1.7^{n-1}$, $F_{n-2} \leq 1.7^{n-2}$

$$F_n \leq 1.7^{n-1} + 1.7^{n-2} = (1.7+1) \cdot 1.7^{n-2} < (2 \cdot 1.7) \cdot 1.7^{n-2} = 1.7^n$$

$\square$

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases:
- $(n=0)$: F_0 & F_3 are both even : 0 & 2
- $(n=1)$: F_1 & F_4 are both odd : 1 & 3

Assume $S(n-1)$ and $S(n-2)$ are true

F_{n-1} & F_{n+2} : both even or both odd

F_{n-2} & F_{n+1} : both even or both odd

By definition, if F_n even then F_{n-1}, F_{n-2} both even or both odd.

If both even then by hypothesis F_{n+2}, F_{n+1} even $\rightarrow F_{n+3}$ even.

If both odd then by hypothesis F_{n+2}, F_{n+1} odd $\rightarrow F_{n+3}$ even.

Similar proof if F_n is odd.

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Tips:

- You can't bridge the gap between $(n-1)$ and n by picking some finite example (e.g. $n=99 \to n=100$)

- Sometimes you will see the inductive step assuming that Claim(n) is known and using it to show Claim($n+1$).

I tend to assume Claim($n-1$) and prove Claim(n).

This is a matter of style. Both are ok.

But I will require that you go from n-1 to n because it will help in the future.

Tips:

- The base case is not always "the smallest number"
  - Sometimes a claim is true only for larger values. $2^n > n^5$
  - Sometimes n just won't rely on "the smallest" instance.

- If Claim(n) relies on multiple "smaller" claims
  make sure you have enough base cases

For all $n \geq 0$, $2^n = 1$ (?!)

What's wrong?

Base case: $n = 0$, $2^0 = 1$ ✓

missed this

$\boxed{\text{for } n \geq 2}$

Hypothesis: assume $2^{n-1} = 1$ and $2^{n-2} = 1$

$$2^{2n-2} = 2^n \cdot 2^{n-2} = 2^{n-1} \cdot 2^{n-1}$$

So $2^n = \dfrac{2^{n-1} \cdot 2^{n-1}}{2^{n-2}} = \dfrac{1 \cdot 1}{1} = 1$

Needed base case for $n = 1$

but $2^1 \neq 1$

$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots$

So if Claim(n) might need two "smaller" claims (e.g $n-1$ & $n-2$),

might we need even more? Yes, but that's often ok.

In fact we can use induction by assuming that
the claim is true for all $k<n$

(technically for all well-defined k, e.g., $0 \le k < n$)

This is called STRONG INDUCTION

FYI - Regular induction and strong induction are actually equivalent
as is proof by smallest counterexample.

Prove: for all $n \geq 2$, n is a product of prime numbers

(possibly more than two of them)

Base case: $n=2 = 2 \cdot 1$ ✓

Hypothesis: "assume that claim is true for all $k < n$"

(but of course we mean $2 \leq k < n$)

If n is prime then $n = n \cdot 1$, done.

Else $n = a \cdot b$ $(a, b \in \mathbb{Z},\ 1 < a < n,\ 1 < b < n)$

Apply hypothesis twice: a and b are each a product of primes

Then $a \cdot b$ is also. $\square$

e.g., $x \cdot y \cdot z$ $\quad p \cdot q \cdot r \cdot s$ $\Big\}$ $n = x \cdot y \cdot z \cdot p \cdot q \cdot r \cdot s$

Proved without knowing exactly what n relied on.

Prove: for all n, $\displaystyle\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$

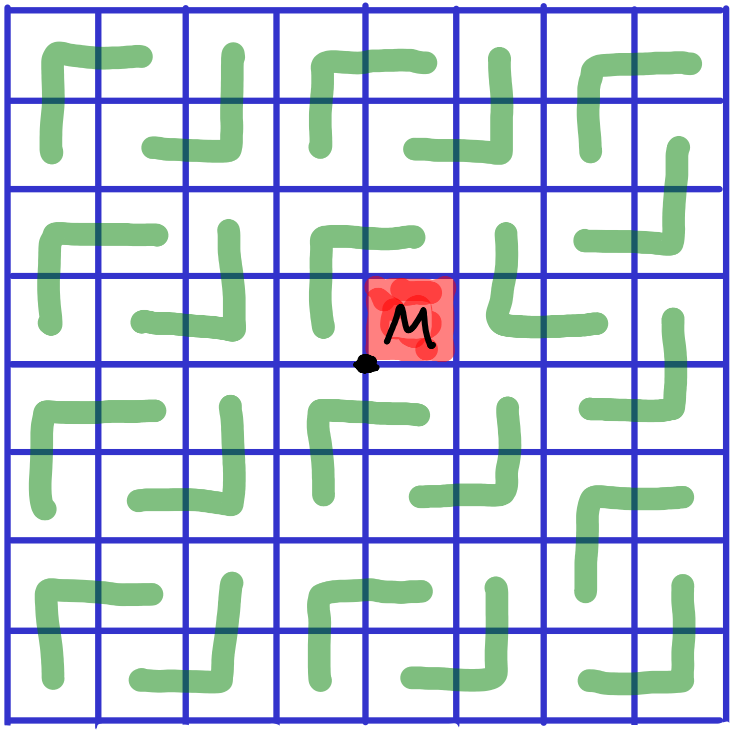
Base case: $n=0$: $\displaystyle\sum_{x=1}^{0} x = 0 = \frac{0(0+1)}{2}$ ✓

Assume $\displaystyle\sum_{x=1}^{n-1} x = \frac{(n-1)((n-1)+1)}{2}$ for $n \geq 1$

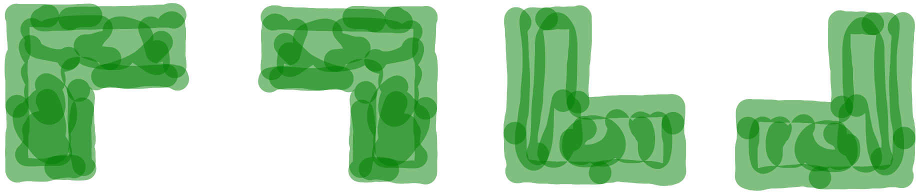
$$\sum_{x=1}^{n} x = n + \underbrace{\sum_{x=1}^{n-1} x} = n + \underbrace{\frac{(n-1)\cdot n}{2}} = \frac{2n + n^2 - n}{2} = \frac{n+n^2}{2} = \frac{n(n+1)}{2}$$

$\square$

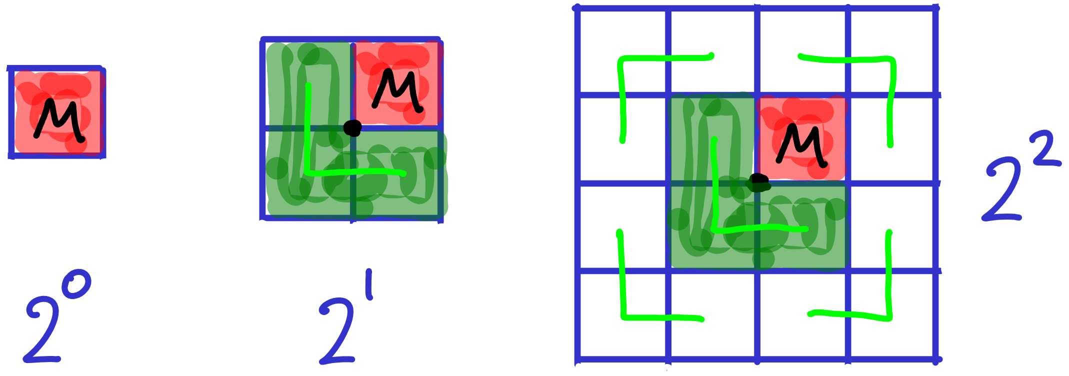
Let $S(n)$ be a grid of $2^n \times 2^n$ squares with one square in the "middle", marked.

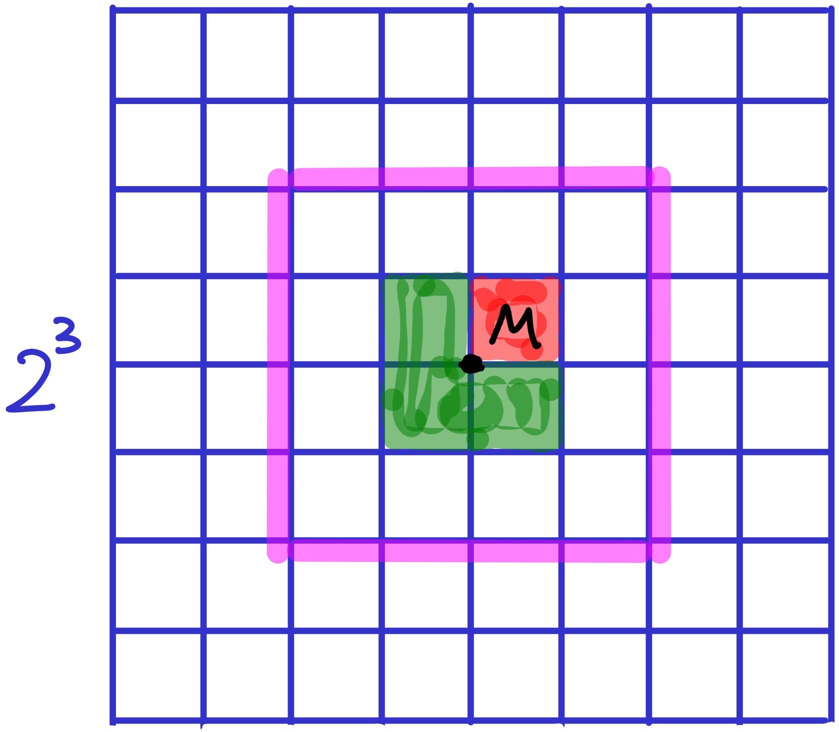


You have L-shaped tiles that you can rotate and place on the grid.



Prove that you can cover the grid with tiles, except for M.





Works for small n (base case)

Hypothesis: $2^{n-1} \times 2^{n-1}$ can be tiled ($n \geq 1$)

But there is no way to use this!

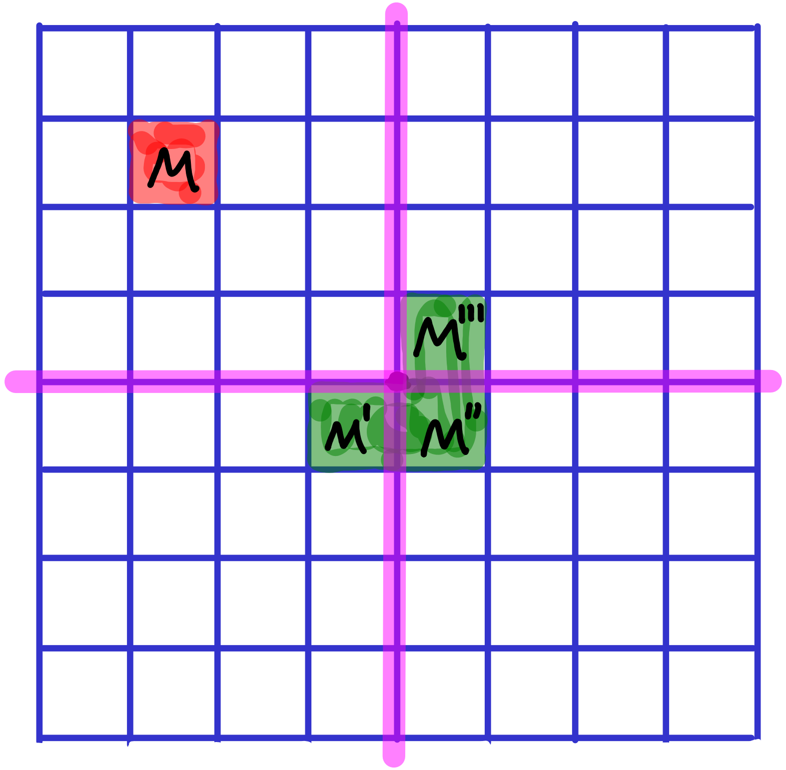
Actual hypothesis = $2^{n-1} \times 2^{n-1}$ grid with M in the middle can be tiled

Hypothesis must match claim

Solution: make problem harder...

Let $S(n)$ be a grid of $2^n \times 2^n$ squares with one square ~~in the "middle"~~ somewhere, marked.

The new problem is harder. We are given less information.



Prove: grid can be covered except for M.

Base case: easy

Hypothesis: $2^{n-1} \times 2^{n-1}$ can be tiled $(n \geq 1)$

Place tile in the middle, avoiding quadrant with M.

By hypothesis, each quadrant can be tiled.

□

Why was it easier to solve a harder problem?

The inductive hypothesis became more powerful.

Let's see another example...

Prove: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2} \leqslant 2$ for all $n \geqslant 1$

Base case: $n=1$: $1 \leqslant 2$ ✓

Assume $\sum_{i=1}^{n-1} \frac{1}{i^2} \leqslant 2$ for $n \geqslant 2$

$$\sum_{i=1}^{n} \frac{1}{i^2} = \frac{1}{n^2} + \sum_{i=1}^{n-1} \frac{1}{i^2} \leqslant \frac{1}{n^2} + 2$$

FAIL

so let's prove something stronger...

Prove: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for all $n \geq 1$

Base case: $n=1$: $1 \leq 2-\frac{1}{1}$ ✓

Assume $\sum_{i=1}^{n-1} \frac{1}{i^2} \leq 2-\frac{1}{n-1}$ for $n \geq 2$

$$\sum_{i=1}^{n} \frac{1}{i^2} = \frac{1}{n^2}+\sum_{i=1}^{n-1} \frac{1}{i^2} \leq \frac{1}{n^2}+2-\frac{1}{n-1} < \frac{1}{n^2}\cdot\frac{n}{n-1}+2-\frac{1}{n-1}$$

$$= \frac{1}{n\cdot(n-1)}+2-\frac{n}{n\cdot(n-1)} = 2-\frac{n-1}{n\cdot(n-1)} = 2-\frac{1}{n}$$

$\square$

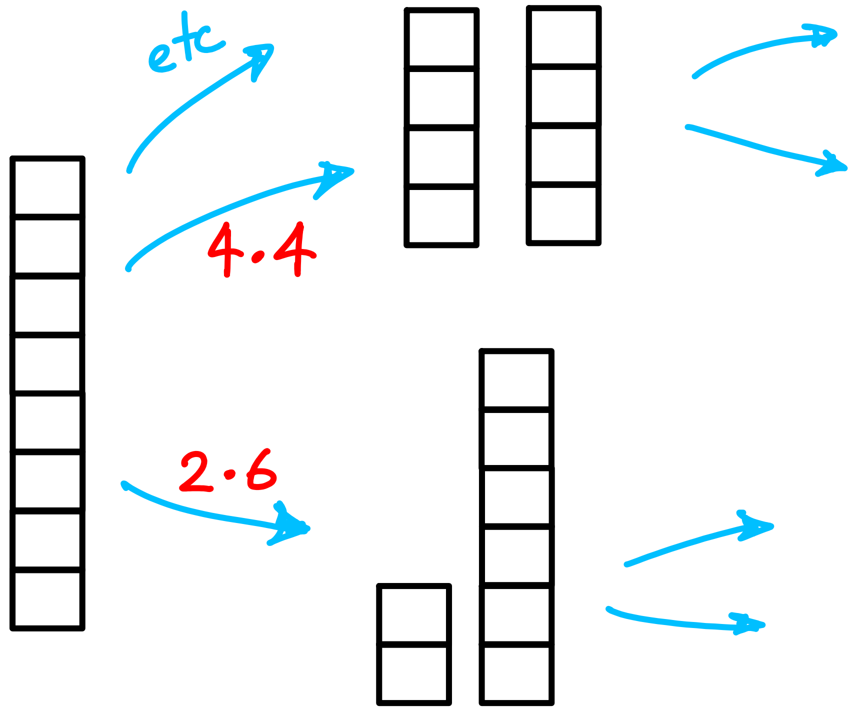
You have a stack of n boxes.

One move: split a stack into 2 new stacks of size a & b.

Reward : $a \cdot b$

Do this until all stacks have size 1.

Try to maximize reward.



Try to balance a,b always?

Product is maximized when equal

Claim: strategy is irrelevant. Reward is always $\frac{n(n-1)}{2}$

Base case: $(n=1)$. Reward $= 0$.

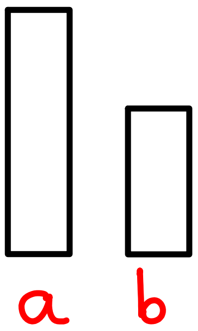
Assume reward $= \frac{k(k-1)}{2}$ if we have a stack of size $k < n$

Consider stack of n boxes. Let 1st move produce stacks $a, b < n$.

$\downarrow$

$a+b$

$\$ = ab$



By hypothesis, future reward $= \frac{a(a-1)}{2} + \frac{b(b-1)}{2}$

$$\text{Total} = \frac{2ab + a^2 - a + b^2 - b}{2} = \frac{(a+b)^2 - (a+b)}{2} = \frac{n^2 - n}{2} = \frac{n(n-1)}{2} \quad \square$$

Let $\begin{cases} T(n) = T\left(\frac{n}{2}\right) + n & \text{for all } n \geq 2 \text{ such that } n = 2^d,\ d \in \mathbb{N} \\ T(1) = 1 \end{cases}$

i.e., $n = 2, 4, 8, 16, \ldots$

Claim: $T(n) = 2n - 1$ for all $n \geq 1$

Base case: $(n=1)$: $T(1) = 1 = 2 \cdot 1 - 1$ ✓

Assume $T(k) = 2k - 1$ for all $1 \leq k < n$ (k : power of 2)

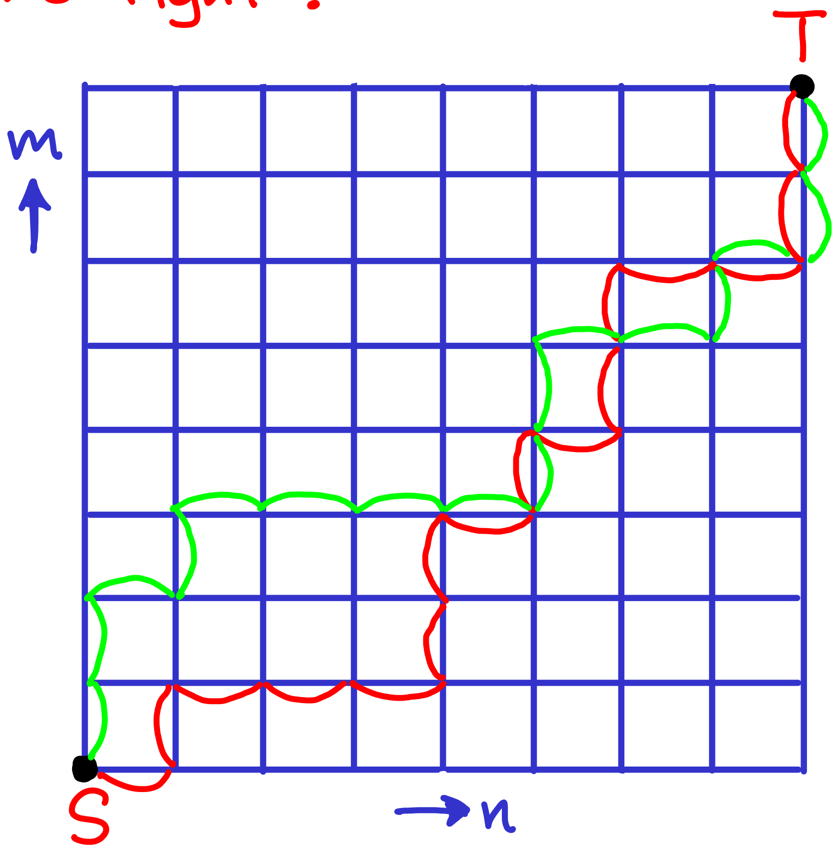
By hypothesis, $T\left(\frac{n}{2}\right) = 2 \cdot \frac{n}{2} - 1 = n - 1$

strong induction, didn't need $T(n-1)$

So $T(n) = (n-1) + n = 2n - 1$

$\square$

How many ways are there to get from $S=(0,0)$ to $T=(n,m)$ if you always take a step up or to the right?

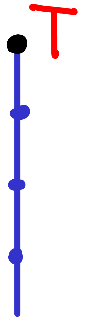


Prove: $W(n,m) = \dfrac{(n+m)!}{n!\,m!}$

Base cases: all grids where $m=0$ or $n=0$

$$W(n,0) = W(0,m) = 1 \quad \text{(fact)}$$

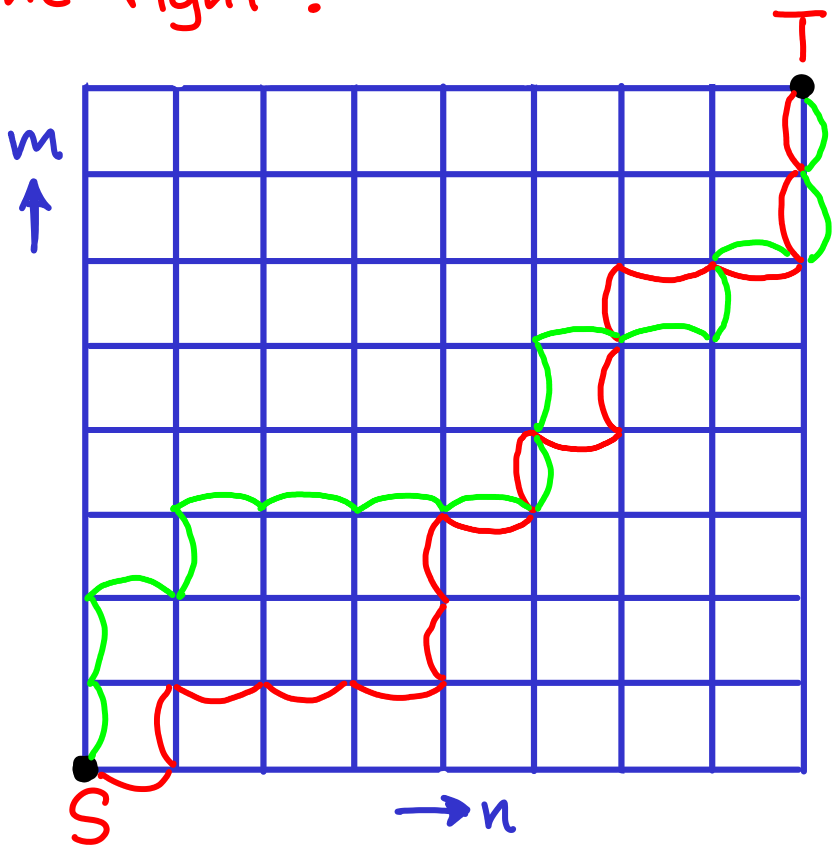
$n=0$



$m=0$

How many ways are there to get from $S=(0,0)$ to $T=(n,m)$ if you always take a step up or to the right?

Prove: $W(n,m) = \frac{(n+m)!}{n!m!}$



Base cases: all grids where $m=0$ or $n=0$

$$W(n,0) = W(0,m) = 1 \quad \text{(fact)}$$

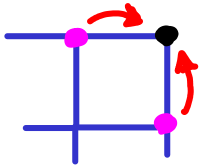
If $m=0$, $\frac{(n+m)!}{n!m!} = \frac{(n+0)!}{n!0!} = 1$ ✓

similar if $n=0$

Hypothesis: if $1 \le x \le n$, $1 \le y \le m$, $x+y < n+m$,

(i.e., if remaining grid is smaller) then $W(x,y) = \frac{(x+y)!}{x!y!}$

$$W(n,m) = W(n-1,m) + W(n,m-1)$$ [must approach from these 2 spots]



If not base case then $n, m \geq 1$.

By hypothesis

$$W(n-1,m) = \frac{((n-1)+m)!}{(n-1)!\,m!} = \frac{(n+m-1)!}{(n-1)!\,m!}$$ ok because $(n-1)+m < n+m$

$$W(n,m-1) = \frac{(n+(m-1))!}{n!\,(m-1)!} = \frac{(n+m-1)!}{n!\,(m-1)!}$$ ok because $n+(m-1) < n+m$

$$W(n,m) = \frac{(n+m-1)!}{(n-1)!\,m!} + \frac{(n+m-1)!}{n!\,(m-1)!} = \frac{n\cdot(n+m-1)!}{n\cdot(n-1)!\,m!} + \frac{m(n+m-1)!}{m\cdot n!\,(m-1)!}$$

$$= \frac{(n+m)\cdot(n+m-1)!}{n!\,m!} = \frac{(n+m)!}{n!\,m!} \quad \square$$

That problem can be solved much faster without induction.

$S \to T$ must have $n+m$ steps.

Of these steps, m go up.

We must choose when to go up: $\binom{n+m}{m} = \dfrac{(n+m)!}{((n+m)-m)!\,m!}$ $\square$

The point was to demonstrate:

- induction with 2 variables
- many base cases: $W(n,0)$ & $W(0,m)$
- how a hypothesis can have creative conditions,

  e.g., $x+y < n+m$

Claim: all horses are the same color.

Rephrase: In every set of $n \geq 1$ horses, all are the same color.

Base case: $(n=1)$ trivially true

Hypothesis: for $k < n$, in every set of k horses, all are the same...

Now look at any set of n horses: $h_1, h_2, h_3, \ldots, h_{n-1}, h_n$

$h_1, h_2, h_3, \ldots, h_{n-1}, h_n$ $\qquad$ $h_1, h_2, h_3, \ldots, h_{n-1}, h_n$

By hypothesis, first $n-1$ are same $\qquad$ and last $n-1$ are same

By overlap, pink = blue ✓ (?!)

What went wrong? → Argument fails for $n=2$

Be careful of general statements