

Suppose you want to prove that $A < C$

but it's complicated to compare directly.

Try to show that $A < B$, and $B < C$

We are specifically interested in comparing functions: $f(n)$ vs $g(n)$

$$f(n) < g(n) \quad ? \checkmark$$

$g(n)$ is an upper bound for $f(n)$

Suppose $n \gg 1$

$$g(n) = 12n^7$$

$$f(n) = n^7 + 3n^5 - 10n^4 + 8\log n$$

$$< n^7 + 3n^5 + 8\log n$$

$$< n^7 + 3n^7 + 8n^7$$

$$= 12n^7 = g(n)$$

EXAGGERATE & SIMPLIFY

Given: complicated $f(n)$

Wanted: simple $g(n)$, such that $f(n) \leq g(n)$ { Secondary goal: minimize $g(n)$ }

$$f(n) = n^7 + \underbrace{3n^5}_{< 3n^7} - \underbrace{n^4}_{< 0} + \underbrace{8 \log n^8}_{= 64 \log n}_{< 64n^7} + \underbrace{3 \log^2 n}_{3n^2}_{< 3n^7} + \underbrace{70 \log_2 n}_{= n^{\log_2 70}}_{< n^{\log_2 128}} + \underbrace{\frac{n^7}{n^{0.1-0.5}}}_{\leq \frac{n^7}{0.5}}_{2n^7}$$

for all n

$$f(n) < n^7 + 3n^7 + 64n^7 + 3n^7 + n^7 + 2n^7 = 74n^7$$

EXAGGERATE & SIMPLIFY

Given: complicated $f(n)$

Wanted: simple $g(n)$, such that $f(n) \leq g(n)$

Secondary goal:
minimize $g(n)$

$$\begin{aligned} f(n) = & n^7 + \underbrace{3n^5}_{< 3n^7} - \underbrace{n^4}_{< 0} + \underbrace{8 \log n^8}_{= 64 \log n} + \underbrace{3 \log^2 n}_{3n^2} + \underbrace{70 \log_2 n}_{= n^{\log_2 70}} + \underbrace{\frac{n^7}{n^{0.1-0.5}}}_{\leq \frac{n^7}{0.5}} \\ & \underbrace{}_{< 64n^7} \quad \underbrace{}_{< 3n^7} \quad \underbrace{\phantom{= n^{\log_2 70}}}_{< n^{\log_2 128}} \quad \underbrace{\phantom{\leq \frac{n^7}{0.5}}}_{< n^7} \end{aligned}$$

(alternative)
if $n > 57$

$$f(n) < n^7 + 3n^7 + 64n^7 + 3n^7 + n^7 + n^7 = 73n^7$$

EXAGGERATE & SIMPLIFY

$$f(n) = n^7 + 3n^5 - n^4 + 8\log n^8 + 3\log^2 n + 70^{\log_2 n} + \frac{n^7}{n^{0.1-0.5}} \leq 73n^7 \quad \text{if } n > 57$$

UNDERESTIMATE & SIMPLIFY

What if we want $f(n) \geq h(n)$ [h : simple] & try to maximize $h(n)$

$$f(n) = n^7 + 3n^5 - n^4 + 8\log n^8 + 3\log^2 n + 70^{\log_2 n} + \frac{n^7}{n^{0.1-0.5}}$$

$$> n^7 - n^4 = \frac{1}{2}n^7 + \frac{1}{2}n^7 - n^4 = \frac{1}{2}n^7 + \left(\frac{1}{2}n^7 - n^4\right) > \frac{1}{2}n^7 \quad \text{if } n \geq 2$$

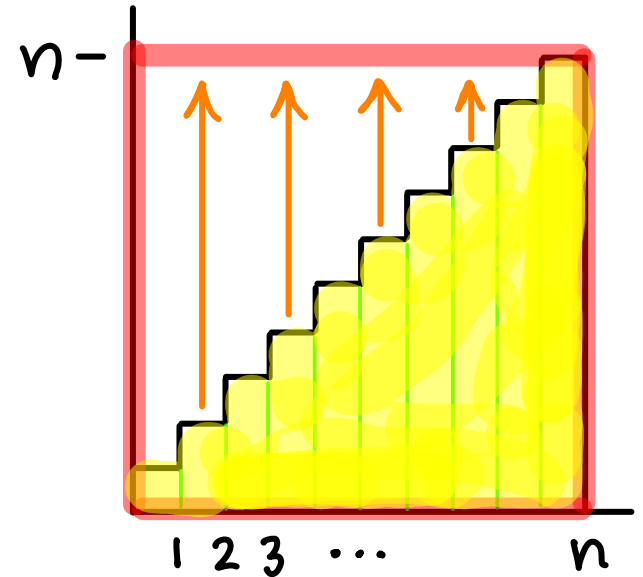
For both simplifications, sometimes you'll need a condition like $n > \dots$

EXAGGERATE & SIMPLIFY

$$f(n) = \sum_{x=1}^n x = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

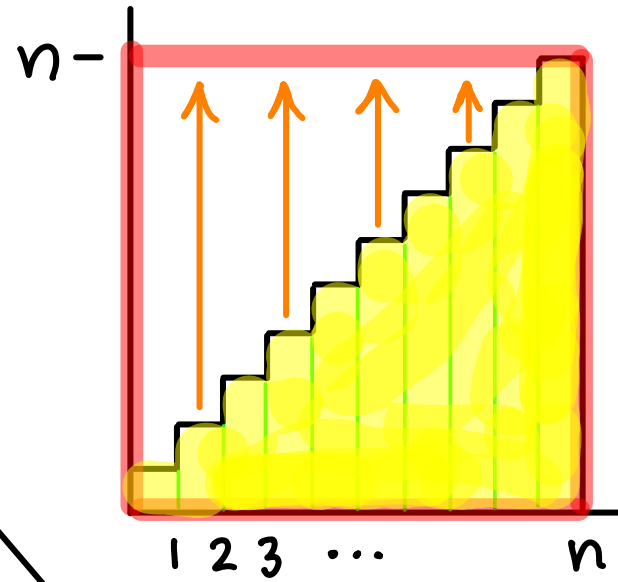
$\downarrow \quad \downarrow \quad \downarrow \quad \quad \downarrow \quad \downarrow$
 $n \quad n \quad n \quad \quad n \quad n$

$$\ll \sum_{x=1}^n n = n^2$$



EXAGGERATE & SIMPLIFY

$$f(n) = \sum_{x=1}^n x \leq \sum_{x=1}^n n = n^2$$



UNDERESTIMATE & SIMPLIFY

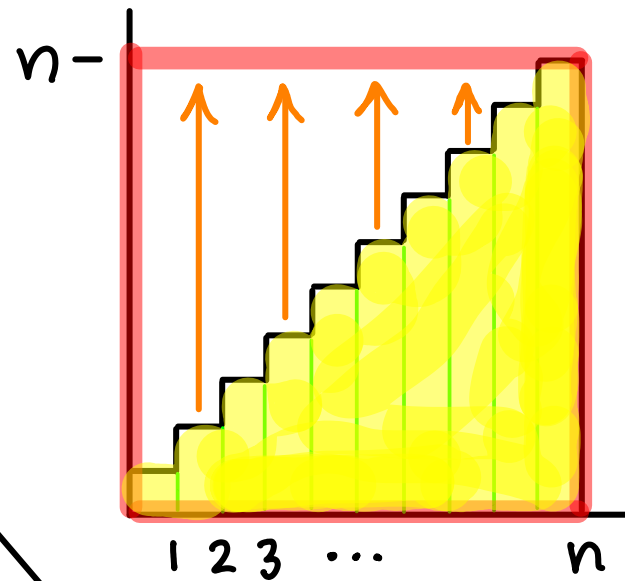
What if we want $f(n) \geq h(n)$ [h: simple]
& try to maximize $h(n)$

$$f(n) = \sum_{x=1}^n x \geq \sum_{x=1}^n 1 = n$$

OK... Can we do better?

EXAGGERATE & SIMPLIFY

$$f(n) = \sum_{x=1}^n x \leq \sum_{x=1}^n n = n^2$$

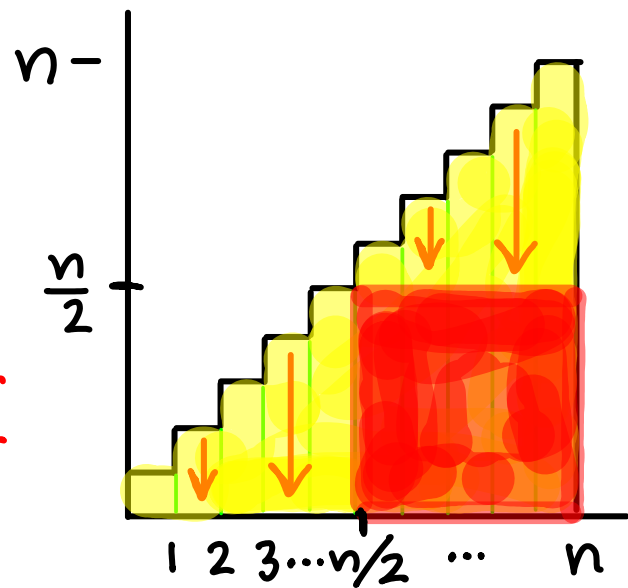


UNDERESTIMATE & SIMPLIFY

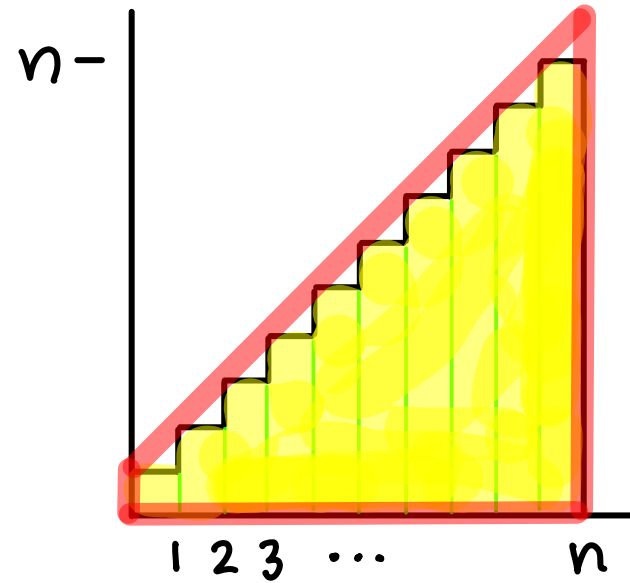
$$f(n) = \sum_{x=1}^n x = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

$\downarrow \quad \downarrow \quad \downarrow$
 $0 \quad 0 \quad 0$

$$\geq \sum_{x=\lceil \frac{n}{2} \rceil}^n x \geq \sum_{x=\lceil \frac{n}{2} \rceil}^n \frac{n}{2} \geq \frac{n^2}{4}$$



$$\frac{n^2}{4} \leq \sum_{x=1}^n x \leq n^2$$



We could have exaggerated less. →

Also, we can show $\sum_{x=1}^n x = \frac{n \cdot (n+1)}{2}$ e.g., by induction

So what was the point?

→ Try $\sum_{x=1}^n x^9$