What is common about these functions?

$5n + \frac{1}{2}n^2 + \frac{1}{100} \cdot n^3$ $\quad 18n^3 - 12$ $\quad 40n^3 + \log n^{10}$

- The dominant term contains n^3

what does this mean? For $n=5$, $\frac{1}{100} \cdot n^3$ doesn't dominate.

$< 5n$ & $\frac{1}{2}n^2$

n^3 dominates for all n larger than some integer.

- All three functions have $50n^3$ as an upper bound, for $n \geq 1$
- All three functions have $\frac{1}{100}n^3$ as a lower bound, for $n \geq 1$

$\boxed{5n + \frac{1}{2}n^2 + \frac{1}{100}\cdot n^3}$ $\boxed{18n^3 - 12}$ $\boxed{40n^3 + \log n^{10}}$

There exist constants $c > 0$, $n_0 > 0$ such that for all $n \geqslant n_0$:

All three functions have cn^3 as an upper bound

(all $\leqslant cn^3$)

There exist constants $d > 0$, $n_1 > 0$ such that for all $n \geqslant n_0$:

All three functions have dn^3 as a lower bound

(all $\geqslant dn^3$)

- All three functions have $50n^3$ as an upper bound, for $n \geqslant 1$
- All three functions have $\frac{1}{100}n^3$ as a lower bound, for $n \geqslant 1$

If there exist constants $c > 0$, $n_0 > 0$ such that for all $n \geq n_0$:

$$f(n) \leq cn^3$$

then we say $f(n) = O(n^3)$

If there exist constants $d > 0$, $n_1 > 0$ such that for all $n \geq n_0$:

$$f(n) \geq dn^3$$

then we say $f(n) = \Omega(n^3)$

If there exist constants $c > 0$, $n_0 > 0$ such that for all $n \geqslant n_0$:

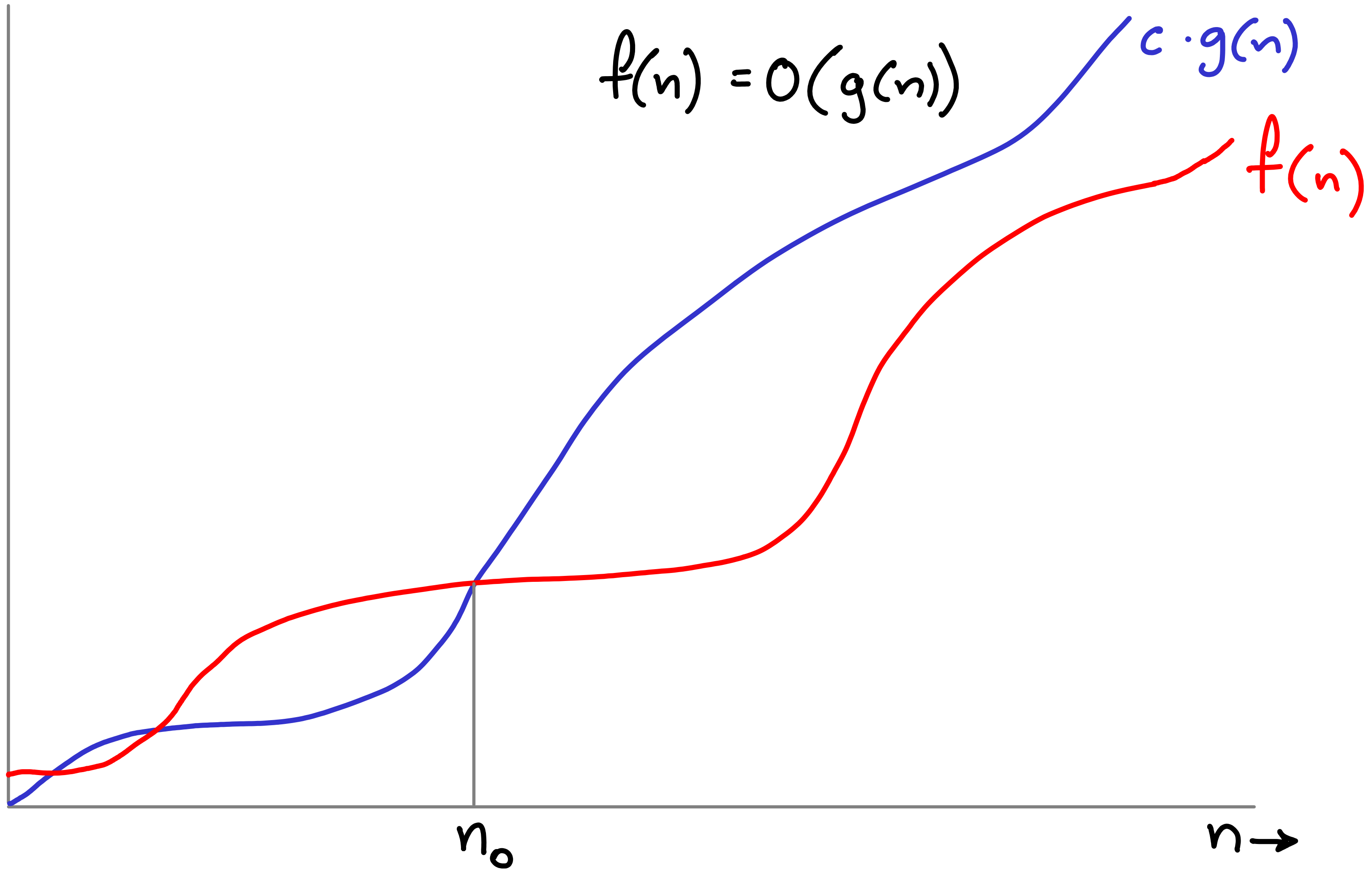
$$f(n) \leq c \cdot g(n)$$

then we say $f(n) = O(g(n))$ Big-O

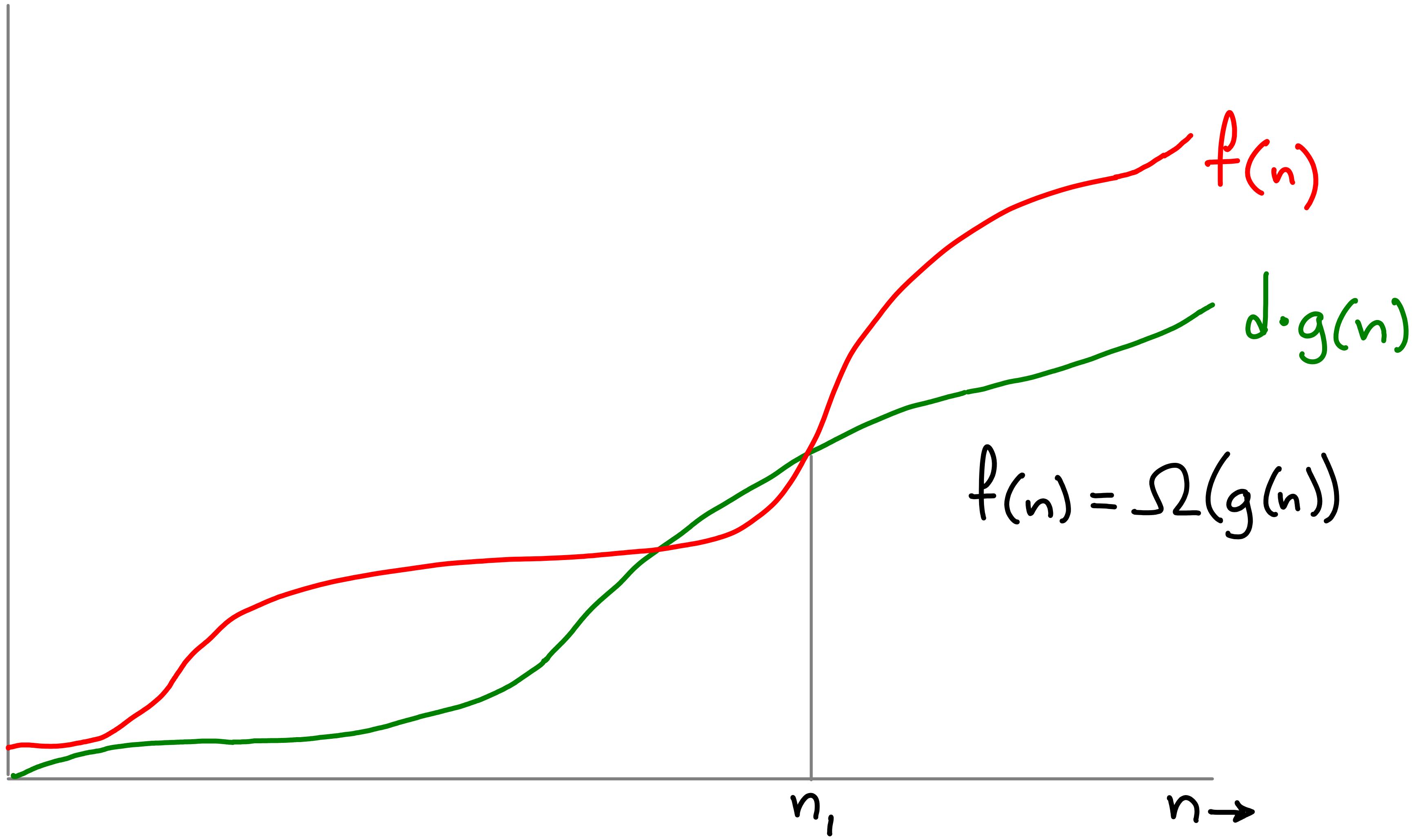
If there exist constants $d > 0$, $n_1 > 0$ such that for all $n \geqslant n_0$:

$$f(n) \geqslant d \cdot g(n)$$

then we say $f(n) = \Omega(g(n))$ Omega

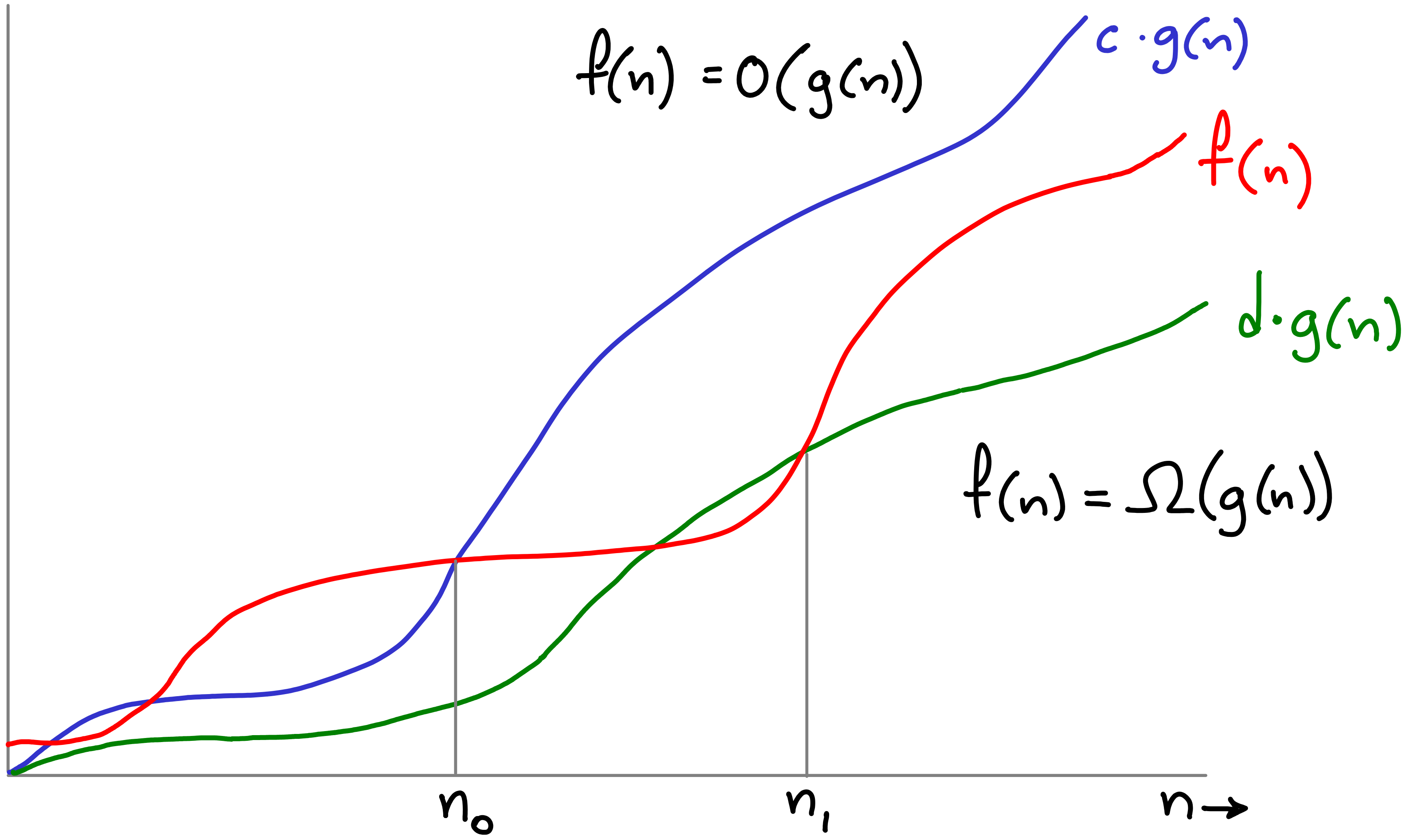


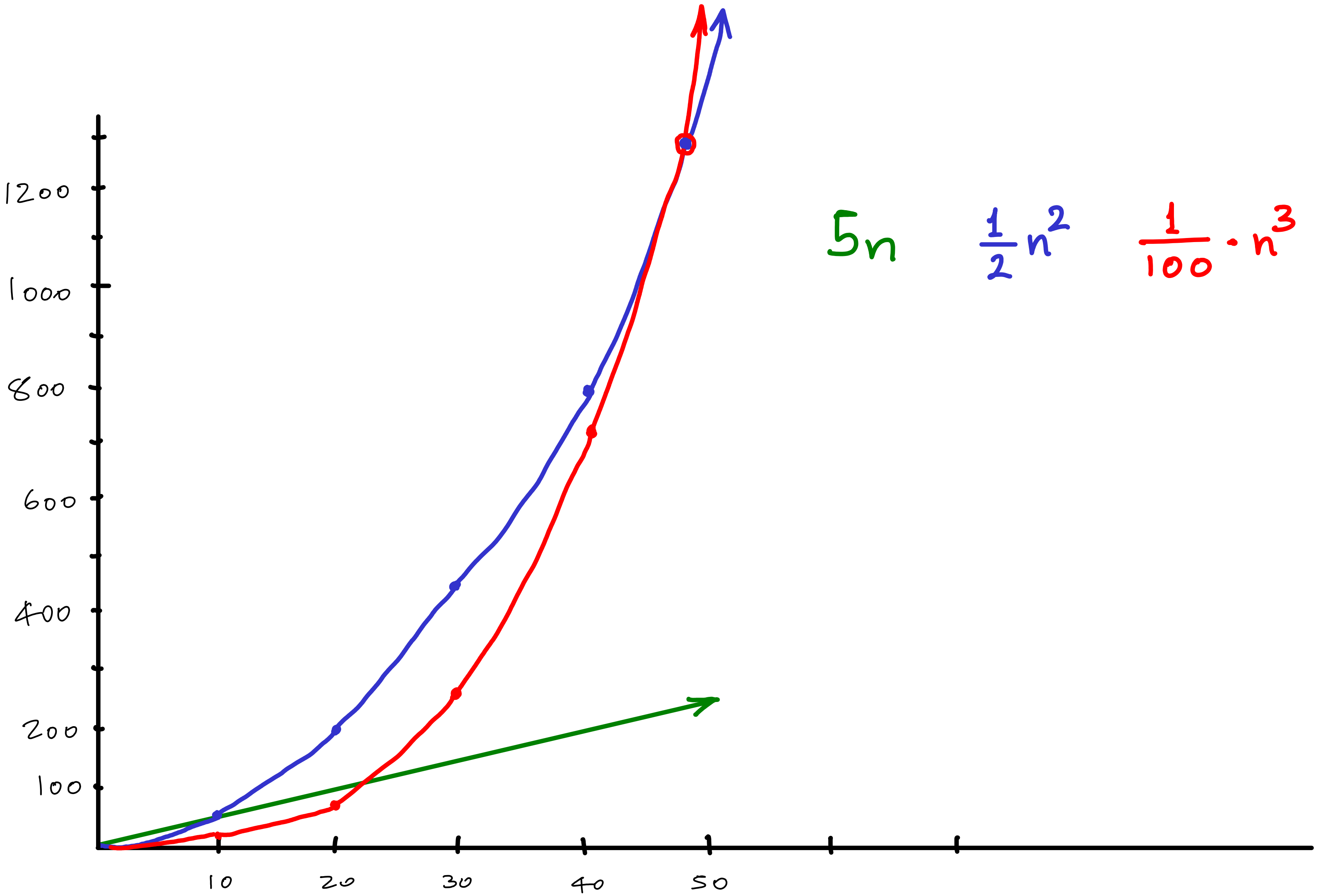
$f(n) = O(g(n))$
$c \cdot g(n)$
$f(n)$
n_0
$n \rightarrow$



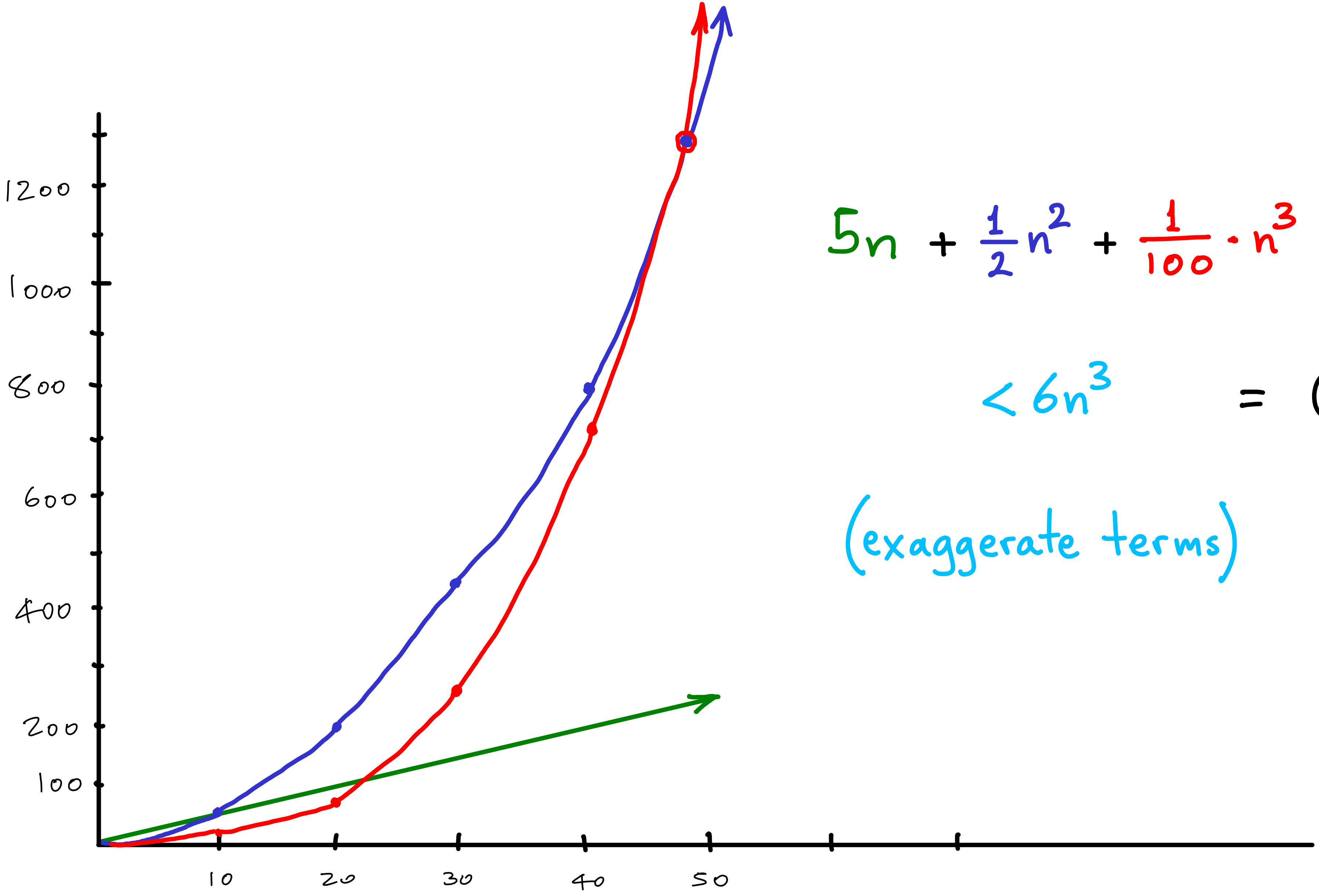
f(n)
d·g(n)
f(n) = Ω(g(n))
n₁
n →

If $f(n) = O(g(n))$ AND $f(n) = \Omega(g(n))$ then $f(n) = \Theta(g(n))$ [Theta]





1200
1000
800
600
400
200
100
10
20
30
40
50
$5n$
$\frac{1}{2}n^2$
$\frac{1}{100}\cdot n^3$

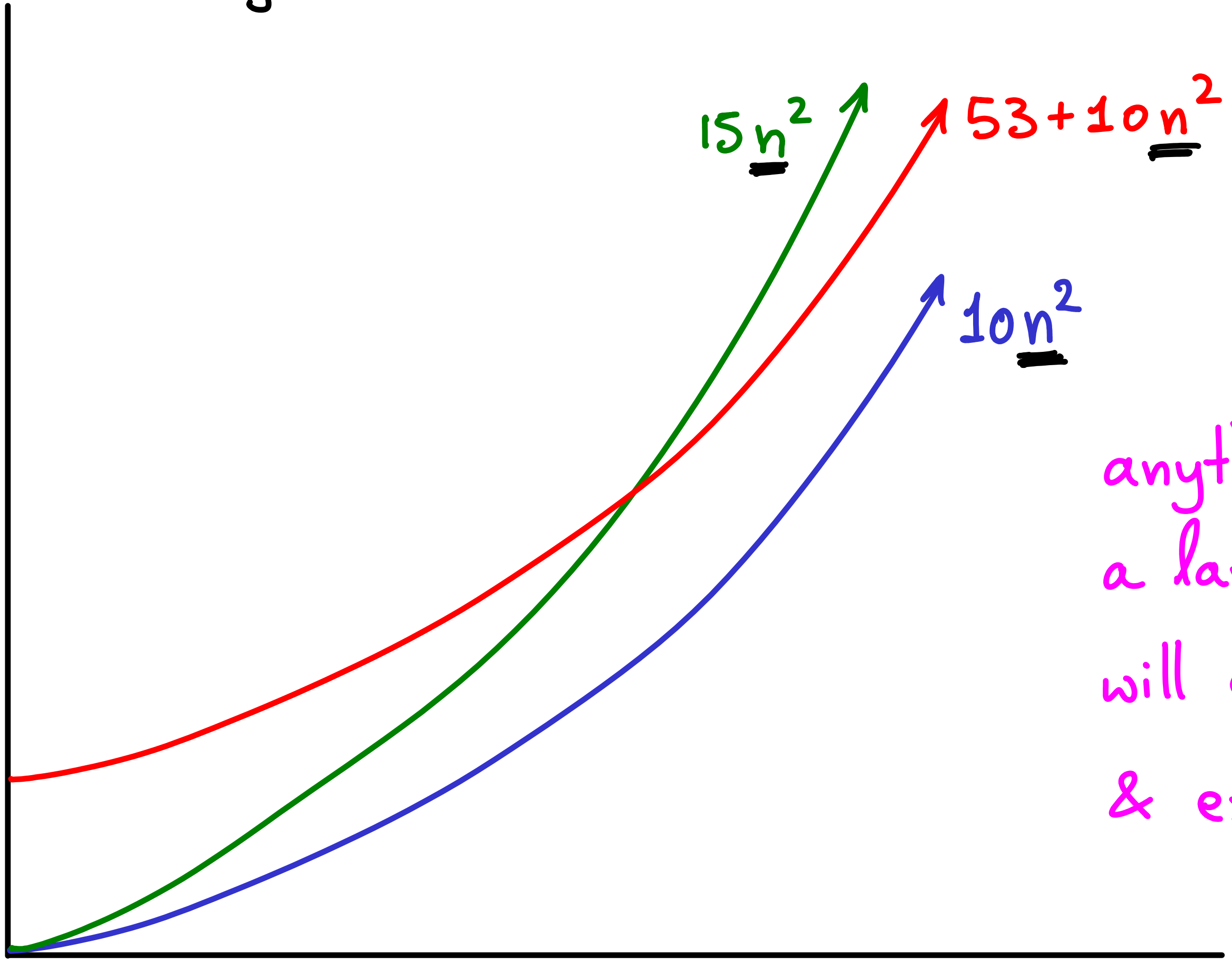


$$5n + \frac{1}{2}n^2 + \frac{1}{100} \cdot n^3$$

$$< 6n^3 \quad = O(n^3)$$

(exaggerate terms)

For large n these are within a constant multiplicative factor



$0.00001 \cdot n^{2.01}$

anything with a larger dominating term will eventually surpass & exceed by a lot.

Polynomials: $a + bn + cn^2 + dn^3 \cdots + \underline{zn}^k = O(\underline{n}^k)$

$a, b, c, d, \ldots, z$: constants

Also assuming one of each term (compare to: $a_1 n + a_2 n + \cdots + a_n n$)

Logarithms: $50 \cdot \log n^3 + \log^{20} n + \underline{n}^{0.1} = O(\underline{n}^{0.1})$

"weaker" than polynomial

Exponential: $100 \cdot n^{50} + \underline{3}^n + 40 \cdot 2^n = O(\underline{3}^n)$

"stronger" than polynomial

Ordering some common functions

$$\underbrace{\qquad\qquad\qquad\qquad}_{}$$ Within each row, subdivide

n^n

$n!$

exponential: k^n $(k>1)$ — 1.1^n, 2^n, 3^n etc

polynomial: n^k — $n^{0.1}$, $\sqrt{n}$, n, $n^{1.1}$, n^2, n^3, etc

powers of logs: $\log^k n = (\log n)^k$ — $\log n$, $\log^2 n$, $\log^3 n$, etc

base doesn't matter!

constants: $1,\ 50,\ 2^{100} = O(1)$

$$\left(50 \cdot \log n + 10\log^5 n + n^{0.1}\right) \cdot \left(100 \cdot n^{50} + 3^n + 40 \cdot 2^n\right) =$$

$$O(n^{0.1}) \cdot O(3^n) \quad = \quad O(n^{0.1} \cdot 3^n)$$

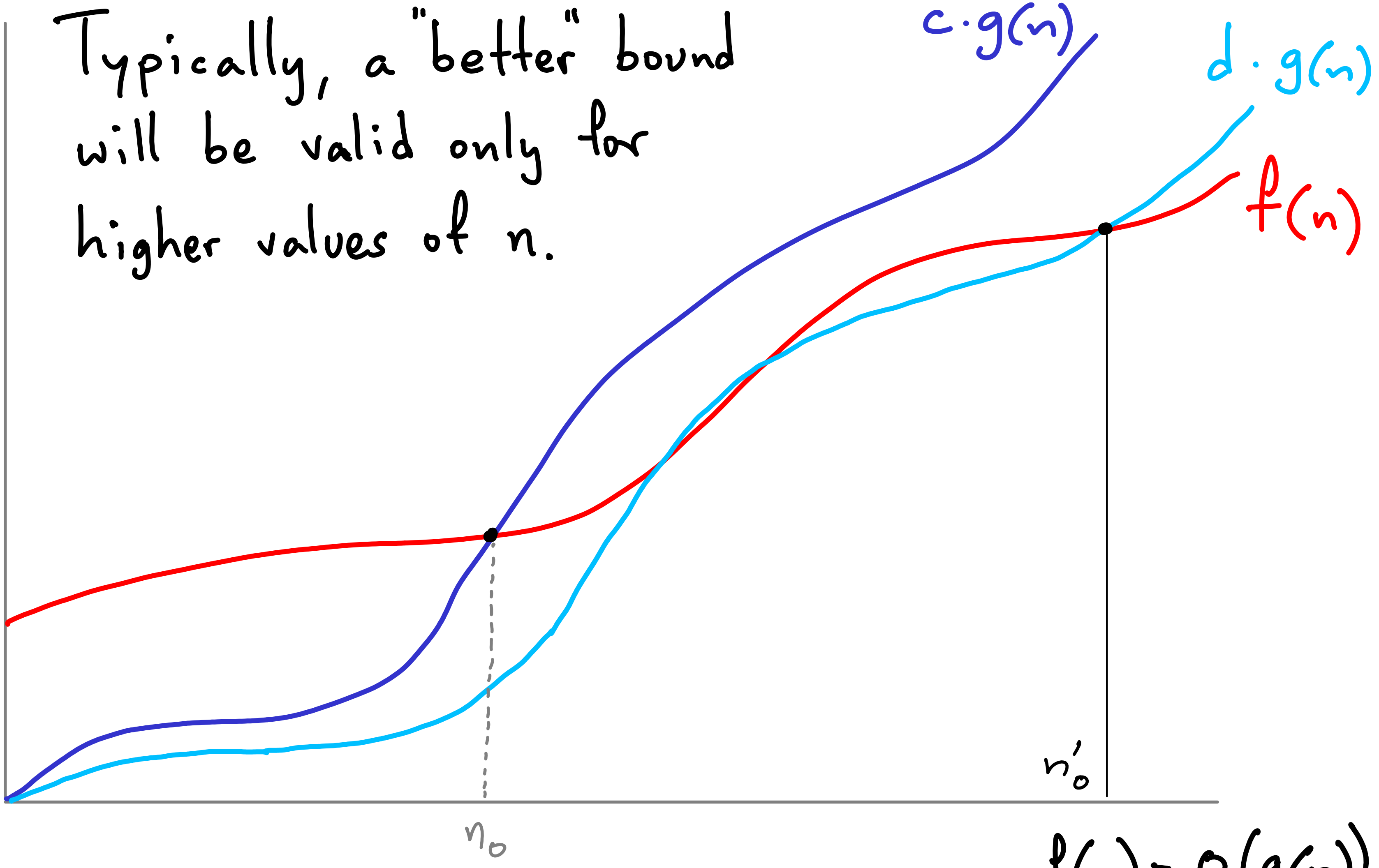
- is $5n^2 = O(n^3)$? yes. but a better answer is $O(n^2)$
- is $n^3 = O(5n^2)$? No.

There is no c such that $n^3 \leq c \cdot 5n^2$

(for all large n)

Also, the 5 doesn't belong in $O(5n^2)$

Typically, a "better" bound will be valid only for higher values of n.



$d < c$

$\downarrow$

better

$f(n) = O(g(n))$

Prove $\frac{1}{2}n^2 + 3n - 10 = \Theta(n^2)$

↳ prove $= O(n^2)$ → find c_1 & n_1 s.t. $\frac{1}{2}n^2 + 3n - 10 \leq c_1 \cdot n^2$ for $n > n_1$

$\frac{1}{2}n^2 + 3n - 10 < 3.5n^2 \Rightarrow c_1 = 3.5$ & $n_1 = 1$ work

(exaggerate & simplify)

↳ prove $= \Omega(n^2)$ → find $\underline{c_2}$ & $\underline{n_2}$ s.t. $\frac{1}{2}n^2 + 3n - 10 \geq \underline{c_2} \cdot n^2$ for $n > \underline{n_2}$

$\frac{1}{2}n^2 + 3n - 10 > \frac{1}{2}n^2 - 10 = \frac{4}{10}n^2 + \underbrace{\left(\frac{1}{10}n^2 - 10\right)}$

(underestimate & simplify) $> \frac{4}{10}n^2$ for $n \geq 10$

$c_2 = 0.4$ & $n_2 = 10$ work

How NOT to prove $f(n) = \frac{1}{2}n^2 + 3n - 10 = O(n^2)$

- Obviously the dominant term is $\frac{1}{2}n^2$, so it's $O(n^2)$

- As $n \to \infty$, the function approaches $\frac{1}{2}n^2$,

  so $f(n) \leq cn^2$, for $c = \frac{1}{2}$

- We need to show $\frac{1}{2}n^2 + 3n - 10 \leq cn^2 \ldots$

$$\frac{\frac{1}{2}\cancel{n^2}}{\cancel{n^2}} + \cancel{\frac{3n}{n^2} - \frac{10}{n^2}} \leq \frac{c\cancel{n^2}}{\cancel{n^2}}$$ so as $n \to \infty$, $\frac{1}{2} \leq c$ $\boxed{\text{NOT OK}}$

Prove $6n^3 \neq \Theta(n^2)$

if it were true, then $\underbrace{c_1 n^2 \leq 6n^3}_{\Omega} \leq c_2 n^2$ for $n \geq n_0$

$\overbrace{6n^3 \leq c_2 n^2}^{O}$

$6n^3 \geq c_1 \cdot n^2 \rightarrow$ trivially true for $n \geq 1$ & $c_1 = 6$

is $6n^3 = O(n^2)$? $\rightarrow$ $6n^3 \leq c_2 n^2$?

$\boxed{\text{NO}}$

$6n \leq c_2$?

$n \leq \frac{c_2}{6}$ } NO.

Whatever constant c_2 we choose, n will eventually surpass it.

Notes:

- $f(n) = O(g(n))$ can be expressed as $f(n) \in O(g(n))$

- Further reading: little-o & little omega (ω)

$$f(n) = o(g(n)) \leftrightarrow f(n) = O(g(n)) \text{ but } f(n) \neq \Theta(g(n))$$

e.g., $n^2 = o(n^3)$ but $0.5n^2 \neq o(n^2)$

$$f(n) = \omega(g(n)) \leftrightarrow f(n) = \Omega(g(n)) \text{ but } f(n) \neq \Theta(g(n))$$

e.g., $n^3 = \omega(n^2)$ but $5n^2 \neq \omega(n^2)$