DS 5220

Supervised Machine Learning and Learning Theory

Alina Oprea
Associate Professor, CCIS
Northeastern University

October 9 2019

Outline

- Joint probability distributions
- Density estimation
  - Kernel density estimation (KDE)
- Naïve Bayes classifier
  - Discrete features
  - Multinomial model

Essential probability concepts

- Marginalization: $P(B) = \sum_{v \in \text{values}(A)} P(B \wedge A = v)$

- Conditional Probability: $P(A \mid B) = \dfrac{P(A \wedge B)}{P(B)}$

- Bayes' Rule: $P(A \mid B) = \dfrac{P(B \mid A) \times P(A)}{P(B)}$

- Independence:

$$
\begin{aligned}
A \perp\!\!\!\perp B \quad &\leftrightarrow \quad P(A \wedge B) = P(A) \times P(B) \\
&\leftrightarrow \quad P(A \mid B) = P(A) \\
A \perp\!\!\!\perp B \mid C \quad &\leftrightarrow \quad P(A \wedge B \mid C) = P(A \mid C) \times P(B \mid C)
\end{aligned}
$$

Prior and Joint Probabilities

- **Prior probability**: degree of belief without any other evidence
- **Joint probability**: matrix of combined probabilities of a set of variables

Russell & Norvig's Alarm Domain: (boolean RVs)

- A world has a specific instantiation of variables:

  $(\text{alarm} \wedge \text{theft} \wedge \neg\text{earthquake})$

- The joint probability is given by:

$P(\text{Alarm}, \text{Theft}) =$

	alarm	¬alarm
theft	0.09	0.01
¬theft	0.1	0.8

Computing Prior Probabilities

	alarm		¬alarm	
	earthquake	¬earthquake	earthquake	¬earthquake
theft	0.01	0.08	0.001	0.009
┐theft	0.01	0.09	0.01	0.79

$$P(alarm) = \sum_{b,e} P(alarm \wedge \text{theft} = b \wedge \mathrm{Earthquake} = e)$$

$$= 0.01 + 0.08 + 0.01 + 0.09 = 0.19$$

$$P(\text{theft}) = \sum_{a,e} P(\mathrm{Alarm} = a \wedge \text{theft} \wedge \mathrm{Earthquake} = e)$$

$$= 0.01 + 0.08 + 0.001 + 0.009 = 0.1$$

The Joint Distribution

Recipe for making a joint distribution of d variables:

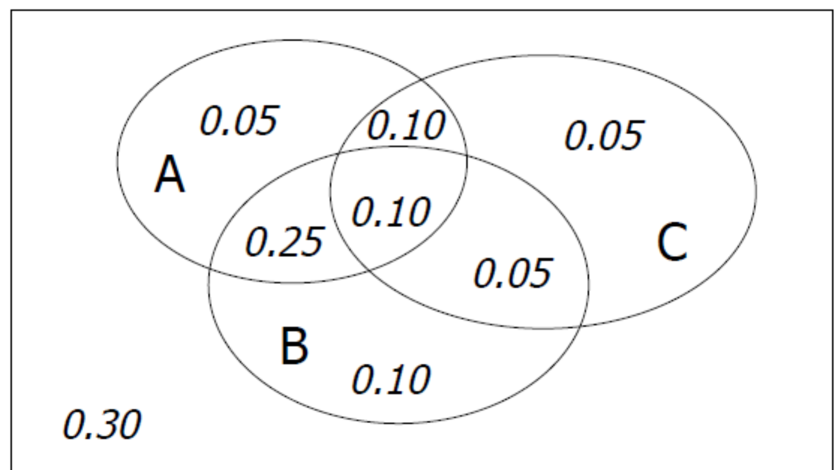
1. Make a truth table listing all combinations of values of your variables (if there are d Boolean variables then the table will have 2^d rows).

2. For each combination of values, say how probable it is.

3. If you subscribe to the axioms of probability, those numbers must sum to 1.

e.g., Boolean variables A, B, C

A	**B**	**C**	**Prob**
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	0.25
1	1	1	0.10



Learning Joint Distributions

Step 1:
Build a JD table for your attributes in which the probabilities are unspecified

A	B	C	Prob
0	0	0	?
0	0	1	?
0	1	0	?
0	1	1	?
1	0	0	?
1	0	1	?
1	1	0	?
1	1	1	?

Step 2:
Then, fill in each row with:

$$\hat{P}(\text{row}) = \frac{\text{records matching row}}{\text{total number of records}}$$

A	B	C	Prob
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	**0.25**
1	1	1	0.10

Fraction of all records in which A and B are true but C is false

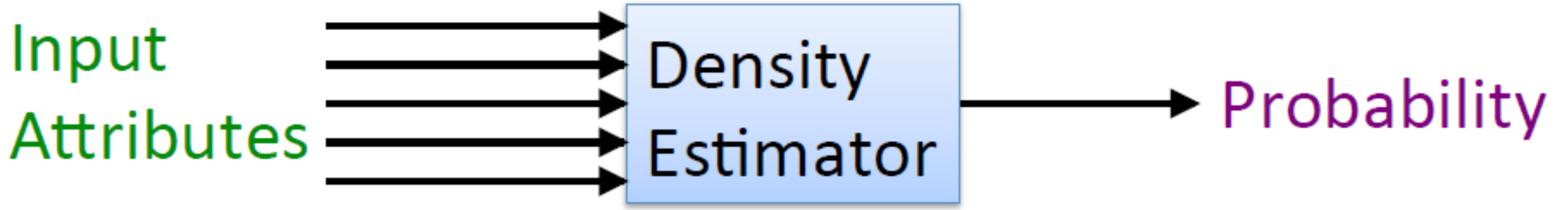
Example – Learning Joint Probability Distribution

This Joint PD was obtained by learning from three attributes in the UCI "Adult" Census Database [Kohavi 1995]

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

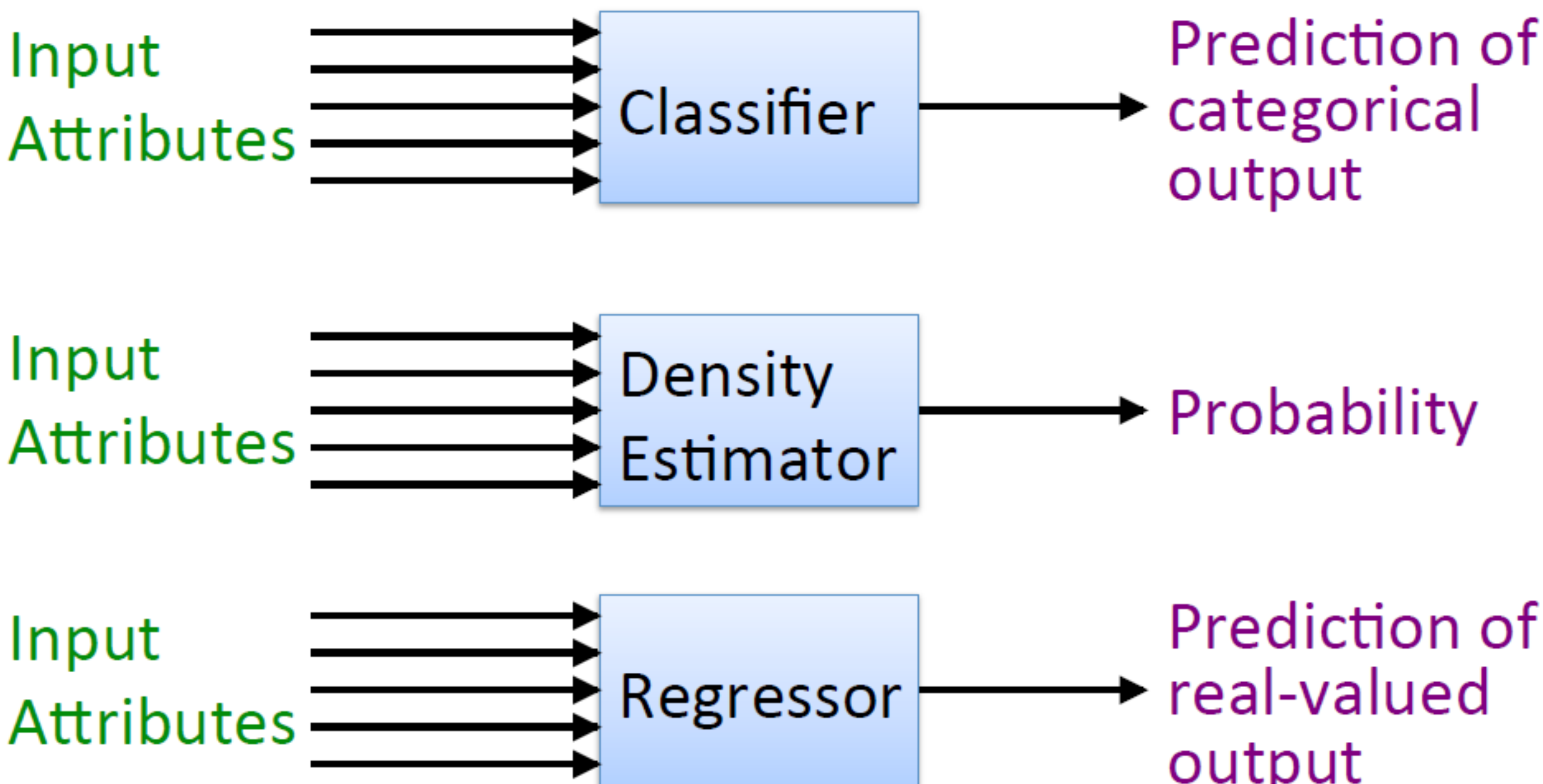
Density Estimation

- Our joint distribution learner is an example of something called **Density Estimation**
- A Density Estimator learns a mapping from a set of attributes to a probability

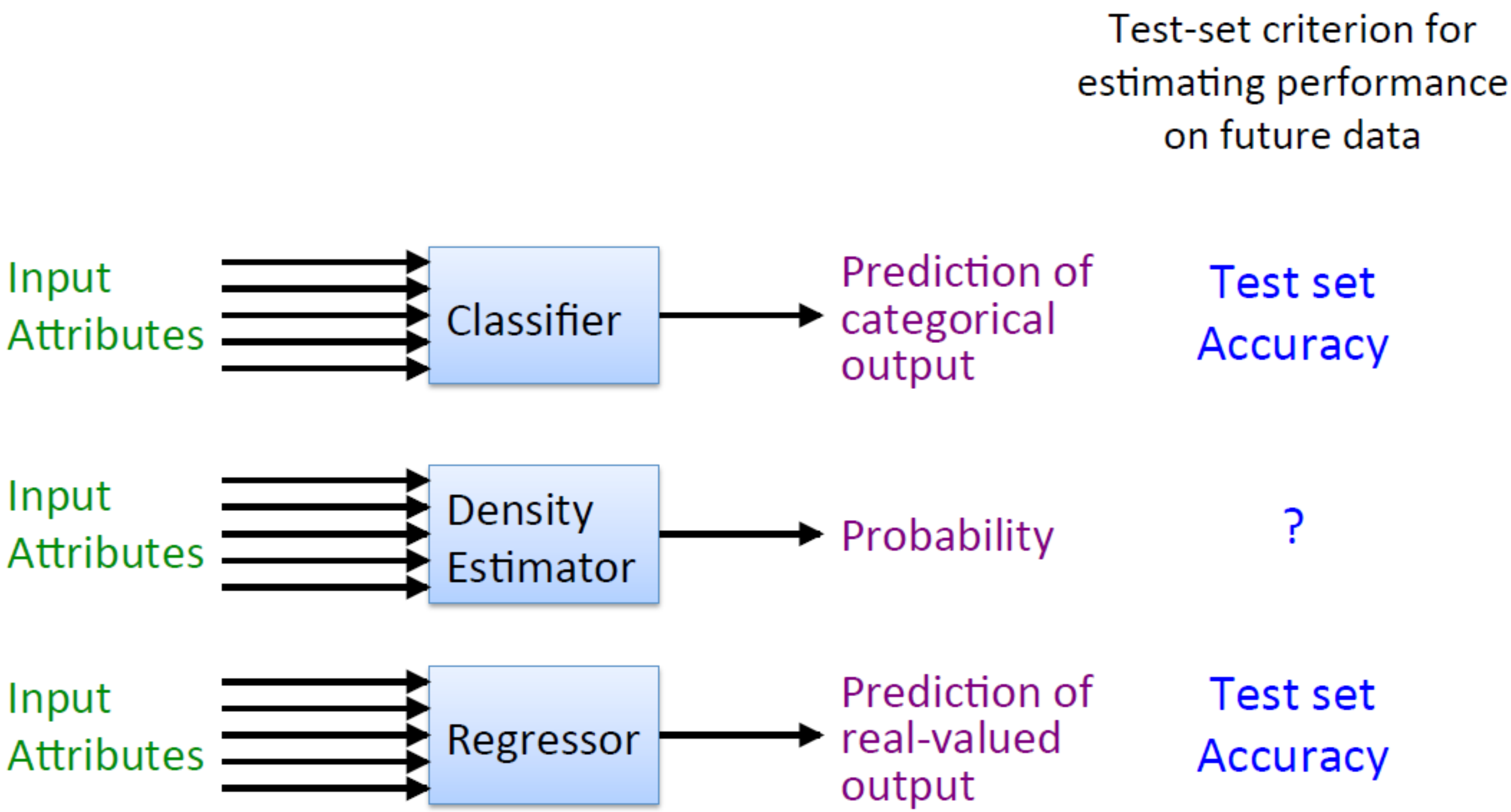


Density Estimation

Compare it against the two other major kinds of models:



Evaluating Density Estimators



Evaluating Density Estimators

- Given a record $\mathbf{x}$, a density estimator *M* can tell you how likely the record is:

$$\hat{P}(\mathbf{x} \mid M)$$

- The density estimator can also tell you how likely the dataset is:
  - Under the assumption that all records were independently generated from the Density Estimator's JD (that is, i.i.d.)

$$\hat{P}(\underbrace{\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_n}_{\text{dataset}} \mid M) = \prod_{i=1}^{n} \hat{P}(\mathbf{x}_i \mid M)$$

Example

From the UCI repository (thanks to Ross Quinlan)

- 192 records in the training set

mpg	modelyear	maker
good	75to78	asia
bad	70to74	america
bad	75to78	europe
bad	70to74	america
bad	70to74	america
bad	70to74	asia
bad	70to74	asia
bad	75to78	america
:	:	:
:	:	:
:	:	:
bad	70to74	america
good	79to83	america
bad	75to78	america
good	79to83	america
bad	75to78	america
good	79to83	america
good	79to83	america
bad	70to74	america
good	75to78	europe
bad	75to78	europe

mpg	modelyear	maker	
bad	70to74	america	0.27551
		asia	0.0255102
		europe	0.0153061
	75to77	america	0.153061
		asia	0.0255102
		europe	0.0357143
	78to83	america	0.0561224
		asia	Never
		europe	Never
good	70to74	america	0.0102041
		asia	0.0306122
		europe	0.0459184
	75to77	america	0.0306122
		asia	0.0408163
		europe	0.0357143
	78to83	america	0.112245
		asia	0.0714286
		europe	0.0357143

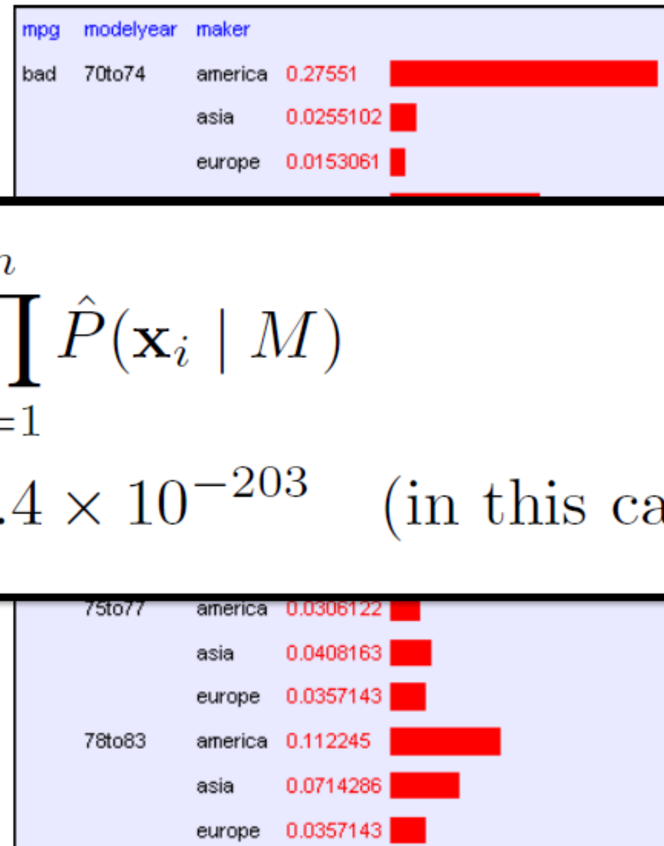
e by Andrew Moore

Example

From the UCI repository (thanks to Ross Quinlan)

- 192 records in the training set

mpg	modelyear	maker
good	75to78	asia
bad	70to74	america
bad	75to78	america
good	79to83	america
good	79to83	america
bad	70to74	america
good	75to78	europe
bad	75to78	europe



$$\hat{P}(\text{dataset} \mid M) = \prod_{i=1}^{n} \hat{P}(\mathbf{x}_i \mid M)$$

$$= 3.4 \times 10^{-203} \quad \text{(in this case)}$$

Slide by Andrew Moore

Log Probabilities

- For decent sized data sets, this product will underflow

$$\hat{P}(\text{dataset} \mid M) = \boxed{\prod_{i=1}^{n}} \hat{P}(\mathbf{x}_i \mid M)$$

- Therefore, since probabilities of datasets get so small, we usually use log probabilities

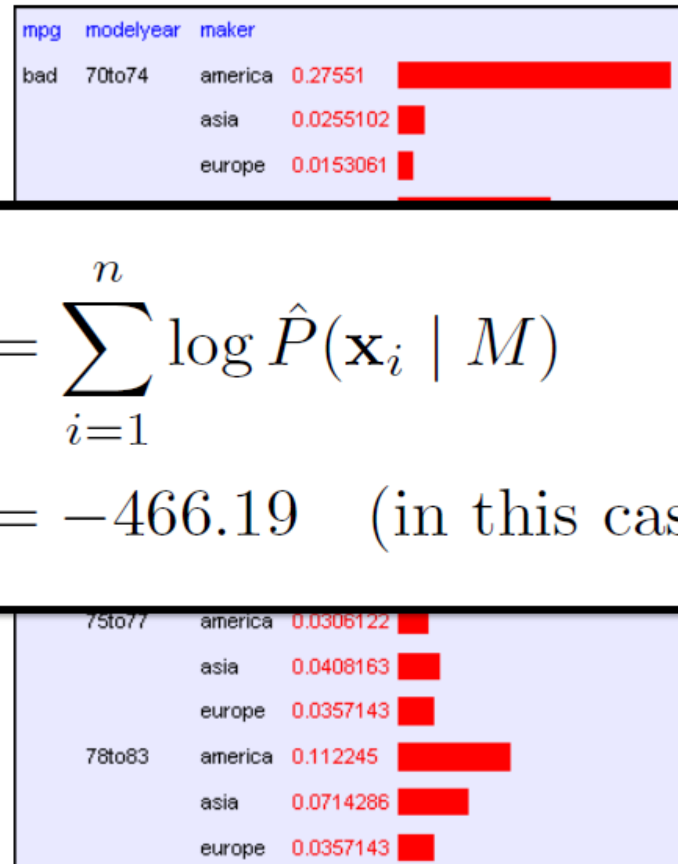
$$\log \hat{P}(\text{dataset} \mid M) = \log \prod_{i=1}^{n} \hat{P}(\mathbf{x}_i \mid M) = \sum_{i=1}^{n} \log \hat{P}(\mathbf{x}_i \mid M)$$

Example

From the UCI repository (thanks to Ross Quinlan)

- 192 records in the training set

mpg	modelyear	maker
good	75to78	asia
bad	70to74	america
bad	75to78	america
good	79to83	america
good	79to83	america
bad	70to74	america
good	75to78	europe
bad	75to78	europe



$$\log \hat{P}(\text{dataset} \mid M) = \sum_{i=1}^{n} \log \hat{P}(\mathbf{x}_i \mid M)$$

$$= -466.19 \quad \text{(in this case)}$$

Slide by Andrew Moore

Evaluation on Test Set

	Set Size	Log likelihood
Training Set	196	-466.1905
Test Set	196	-614.6157

- An independent test set with 196 cars has a much worse log-likelihood
  - Actually it's a billion quintillion quintillion quintillion quintillion times less likely

- Density estimators can overfit...

  ...and the full joint density estimator is the overfittiest of them all!

Overfitting

If this ever happens, the joint PDE learns there are certain combinations that are impossible

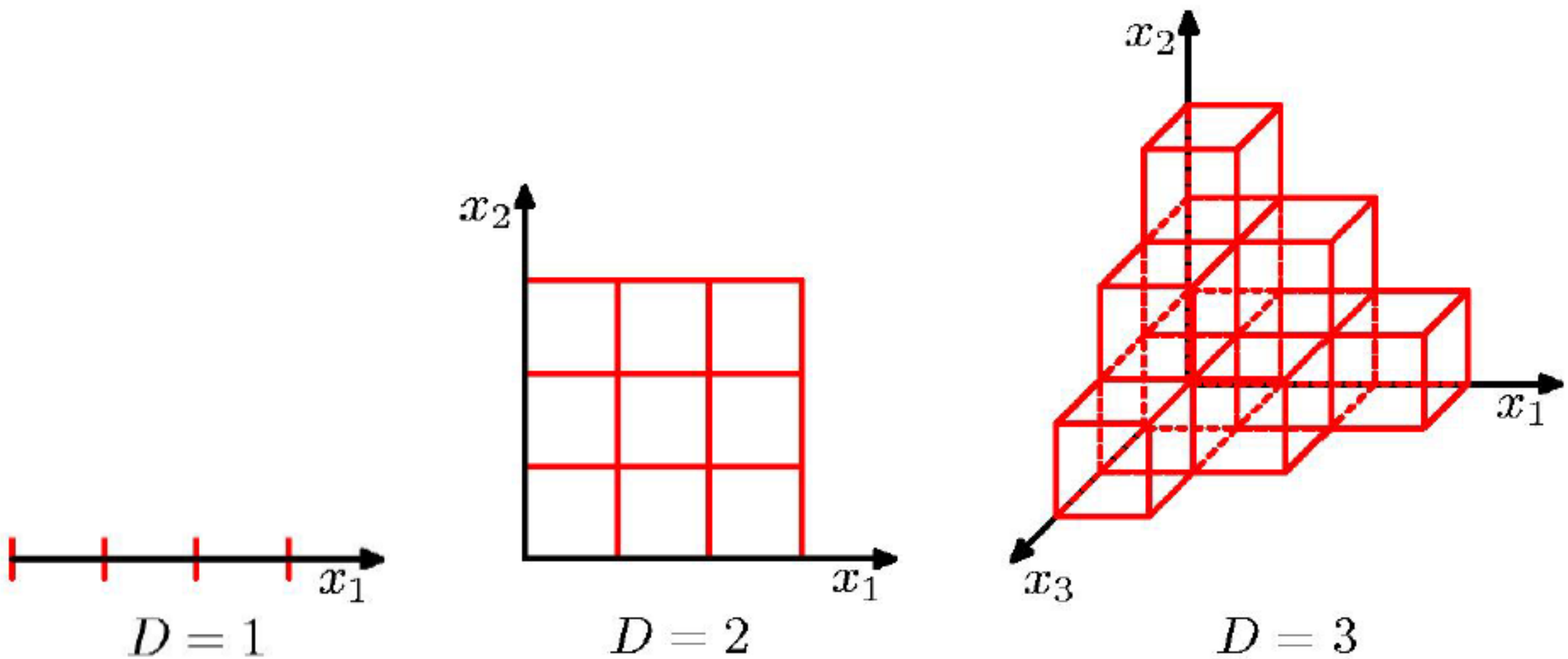
mpg	modelyear	maker		
bad	70to74	america	0.27551	
		asia	0.0255102	
		europe	0.0153061	
	75to77	america	0.153061	
		asia	0.0255102	
		europe	0.0357143	
	78to83	america	0.0561224	
		asia	Never	
		europe	Never	
good	70to74	america	0.0102041	
		asia	0.0306122	

$$\log \hat{P}(\text{dataset} \mid M) = \sum_{i=1}^{n} \log \hat{P}(\mathbf{x}_i \mid M)$$

$$= -\infty \quad \text{if for any } i,\ \hat{P}(\mathbf{x}_i \mid M) = 0$$

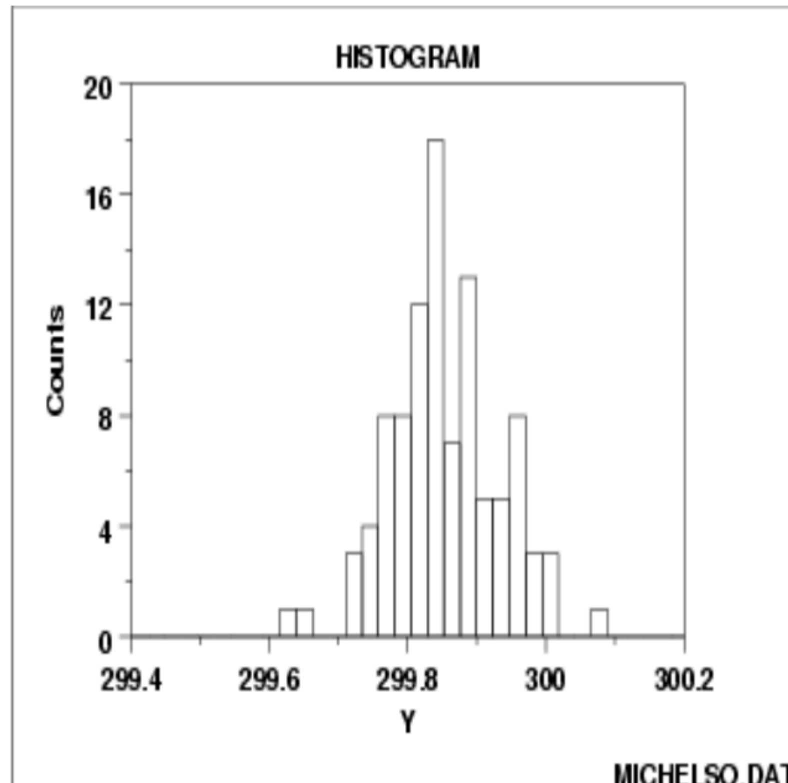
de by Andrew Moore

Curse of Dimensionality



Histograms

- Most common form of non-parametric density estimation
  - Split data range into equal-sized bins
  - For each bin, count the # of data points that fall into the bin
  - Y axis: frequency (e.g. counts for each bin)
  - X axis: values of the variable
- The histogram can illustrate features related to the distribution of the data
  - Center (i.e., the location)
  - Spread (i.e., the scale)
  - Skewness
  - Presence of outliers



Issues with Histograms

- Need to select the two parameters: starting position of bin and width
  - For small datasets, the shape of the histogram looks different when parameters change
- Curse of dimensionality
  - Number of bins grows exponentially with the number of dimensions
  - In high dimensions, a very large number of examples is required; otherwise most of the bins will be empty

Unsuitable for most practical applications expect for quick visualization in one or two dimensions

Kernel Density Estimation

A kernel function K is a function such that...

- $K(x) \geq 0$ for all $-\infty < x < \infty$
- $K(-x) = K(x)$
- $\int_{-\infty}^{\infty} K(x) \mathrm{d}x = 1$

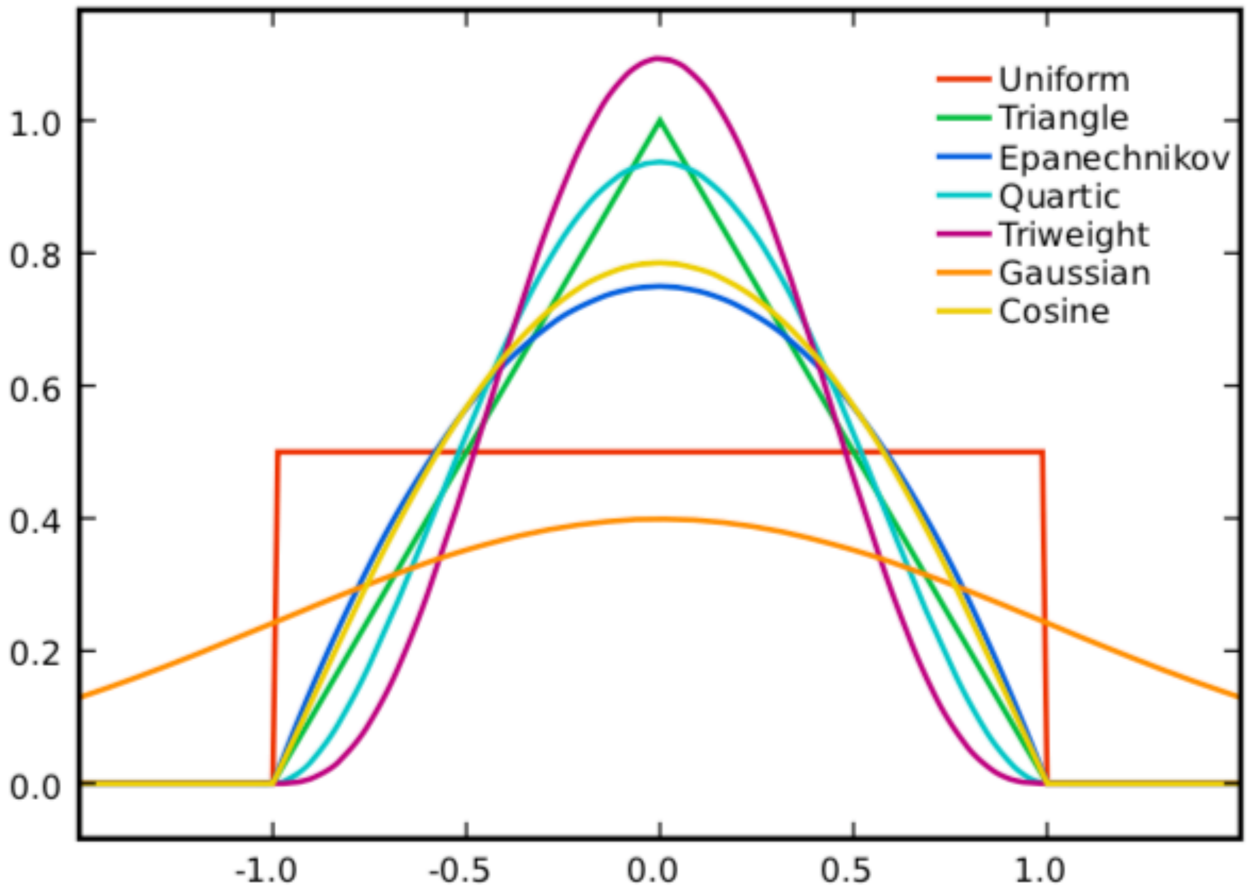
A simple example is the uniform (or box) kernel:

$$K(x) = \begin{cases} 1 & \text{if } -1/2 \leq x < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Another popular kernel function is the Normal kernel (pdf) with $\mu = 0$ and σ fixed at some constant:

$$K(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

Kernel Visualization



From http://upload.wikimedia.org/wikipedia/commons/4/47/Kernels.svg

Kernel Density Estimate

$$K_h^{(x_i)}(x) = \frac{1}{h} K\left(\frac{x - x_i}{h}\right)$$

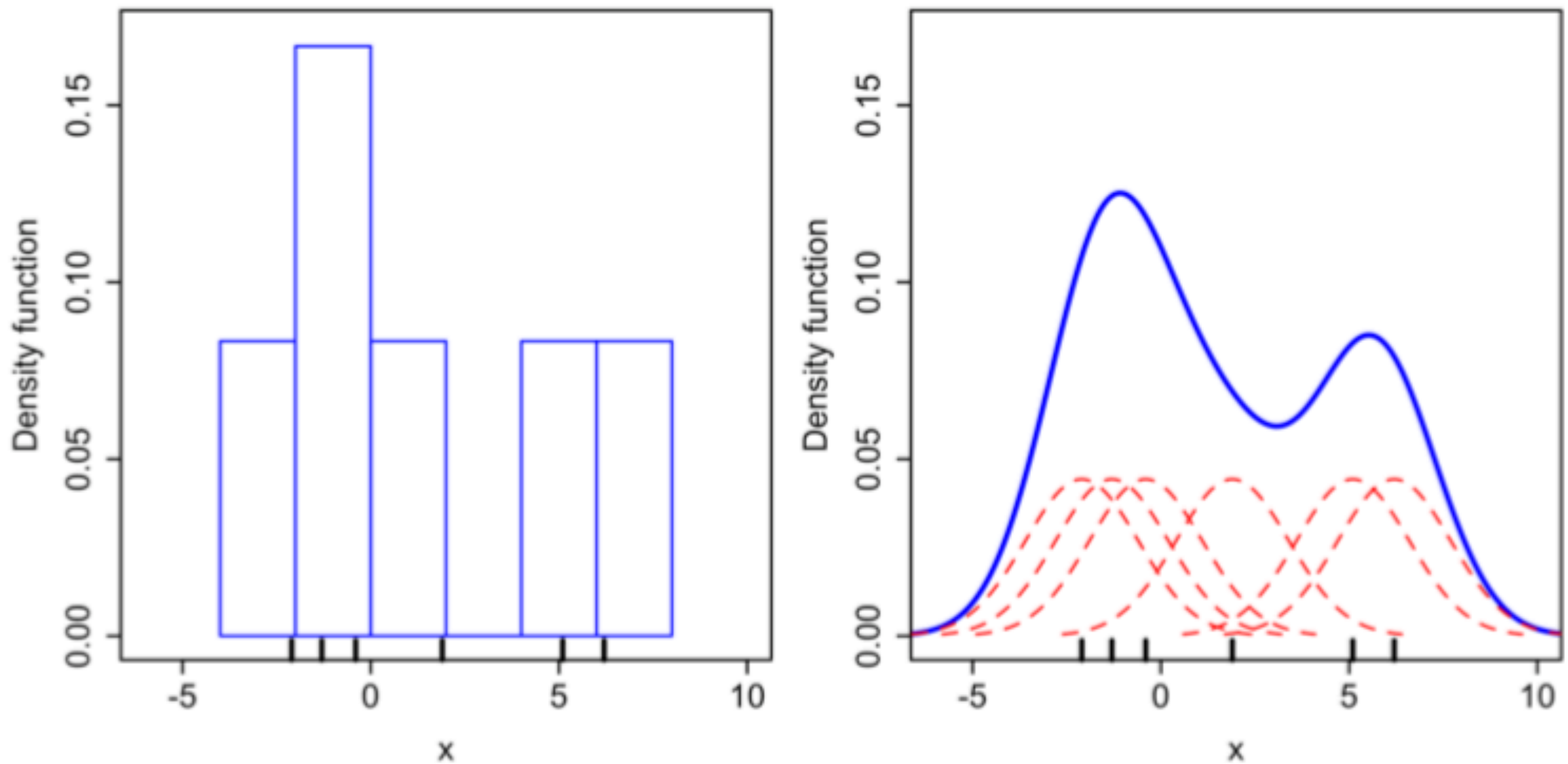
Scaled and centered kernel

Given a random sample $x_i \overset{\text{iid}}{\sim} f(x)$, the kernel density estimate of f is

$$\begin{aligned}\hat{f}(x) &= \frac{1}{n} \sum_{i=1}^{n} K_h^{(x_i)}(x) \\ &= \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)\end{aligned}$$

where h is now referred to as the bandwidth (instead of bin width).

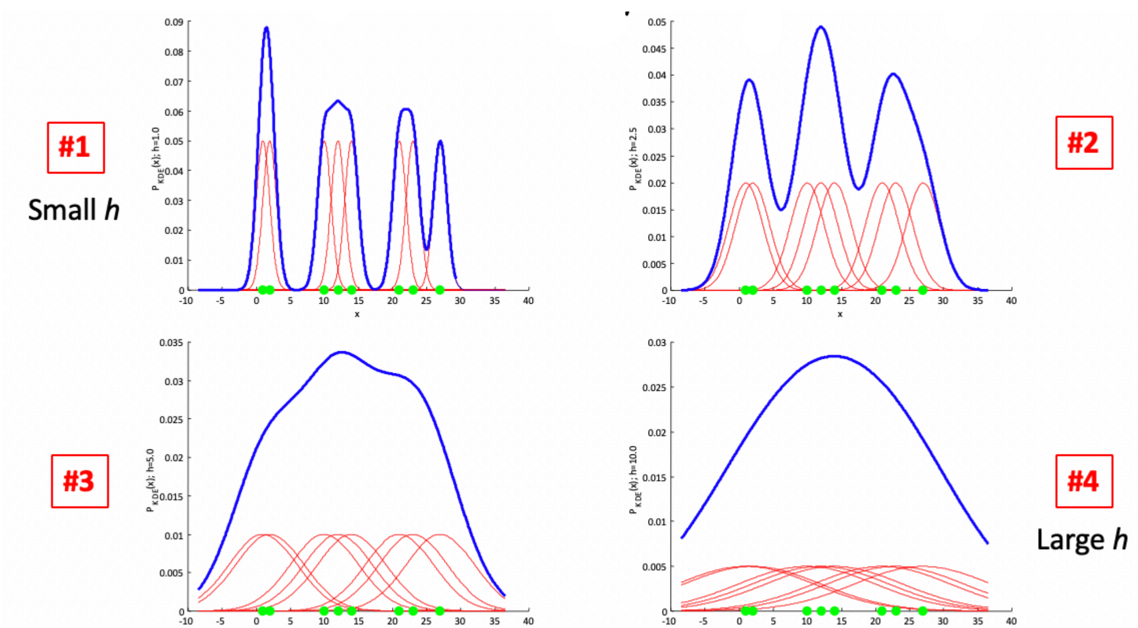
Kernel Density Estimate: Visualization

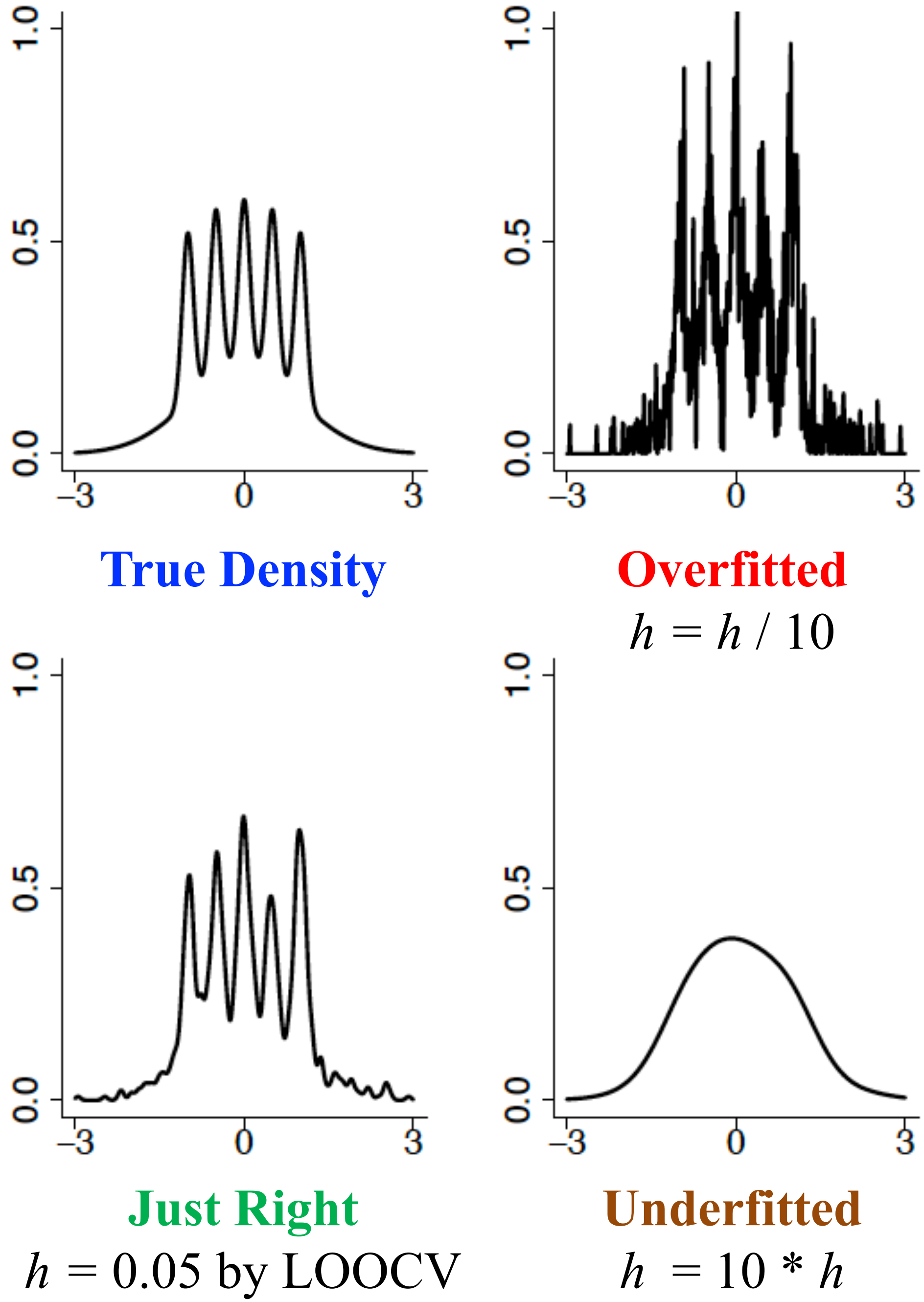


From http://en.wikipedia.org/wiki/Kernel_density_estimation

Bandwidth Selection

- The problem of choosing $h = \sigma$ is crucial in density estimation
- Small bandwidth: over-fitting
- Large bandwidth: can mask the data structure





- KDE based on n = 1;000 draws
- LOOCV = leave one out cross-validation.

Naïve Bayes Classifier
Another Method for Density Estimation

Idea: Use the training data to estimate

$P(X \mid Y)$ and $P(Y)$.

Then, use Bayes rule to infer $P(Y|X_{\text{new}})$ for new data

---

Easy to estimate from data

Impractical, but necessary

$$\mathrm{P}[Y = k|X = x] = \frac{\boxed{\mathrm{P}[Y = k]}\,\boxed{\mathrm{P}[X_1 = x_1 \wedge \cdots \wedge X_d = x_d|Y = k]}}{\boxed{\mathrm{P}[X_1 = x_1 \wedge \cdots \wedge X_d = x_d]}}$$

Unnecessary, as it turns out

- Recall that estimating the joint probability distribution $P(X_1, X_2, \ldots, X_d \mid Y)$ is not practical

Naïve Bayes Classifier

Problem: estimating the joint PD or CPD isn't practical

– Severely overfits, as we saw before

However, if we make the assumption that the attributes are independent given the class label, estimation is easy!

$$\mathrm{P}[X_1 = x_1 \wedge \cdots \wedge X_d = x_d | Y = k] = \prod_{j=1}^{d} P[X_j = x_j | Y = k]$$

- In other words, we assume all attributes are *conditionally independent* given Y
- Often this assumption is violated in practice, but more on that later…

Training Naïve Bayes

Estimate $P[X_j = x_j | Y = k]$ and $P[Y = k]$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
rainy	cold	high	strong	warm	change	*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P}(\mathrm{play}) = ?$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \mathrm{play}) = ?$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \mathrm{play}) = ?$

$\ldots$

$\mathrm{P}(\neg\mathrm{play}) = ?$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \neg\mathrm{play}) = ?$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \neg\mathrm{play}) = ?$

$\ldots$

Training Naïve Bayes

Estimate $P[X_j = x_j | Y = k]$ and $P[Y = k]$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
rainy	cold	high	strong	warm	change	*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P}(\mathrm{play}) =$ 3/4

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \mathrm{play}) = ?$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \mathrm{play}) = ?$

$\ldots$

$\mathrm{P}(\neg\mathrm{play}) =$ 1/4

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \neg\mathrm{play}) = ?$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \neg\mathrm{play}) = ?$

$\ldots$

Training Naïve Bayes

Estimate $P[X_j = x_j | Y = k]$ and $P[Y = k]$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny						*yes*
sunny						*yes*
rainy	cold	high	strong	warm	change	*no*
sunny						*yes*

$\mathrm{P(play)} = 3/4$

$\mathrm{P(Sky = sunny \mid play)} = 1$

$\mathrm{P(Humid = high \mid play)} = ?$

$\ldots$

$\mathrm{P(\neg play)} = 1/4$

$\mathrm{P(Sky = sunny \mid \neg play)} = ?$

$\mathrm{P(Humid = high \mid \neg play)} = ?$

$\ldots$

Training Naïve Bayes

Estimate $P[X_j = x_j | Y = k]$ and $P[Y = k]$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
rainy						*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P(play)} = 3/4$

$\mathrm{P(Sky = sunny \mid play)} = 1$

$\mathrm{P(Humid = high \mid play)} = ?$

$\ldots$

$\mathrm{P(\neg play)} = 1/4$

$\mathrm{P(Sky = sunny \mid \neg play)} = 0$

$\mathrm{P(Humid = high \mid \neg play)} = ?$

$\ldots$

Training Naïve Bayes

Estimate $P[X_j = x_j | Y = k]$ and $P[Y = k]$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
		normal				*yes*
		high				*yes*
rainy	cold	high	strong	warm	change	*no*
		high				*yes*

$\mathrm{P}(\mathrm{play}) = 3/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \mathrm{play}) = 1$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \mathrm{play}) =$ 2/3

...

$\mathrm{P}(\neg\mathrm{play}) = 1/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \neg\mathrm{play}) = 0$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \neg\mathrm{play}) = ?$

...

Training Naïve Bayes

Estimate $P[X_j = x_j | Y = k]$ and $P[Y = k]$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
		high				*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P}(\mathrm{play}) = 3/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \mathrm{play}) = 1$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \mathrm{play}) = 2/3$

...

$\mathrm{P}(\neg\mathrm{play}) = 1/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \neg\mathrm{play}) = 0$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \neg\mathrm{play}) = 1$

...

Laplace Smoothing

- Notice that some probabilities estimated by counting might be zero
  - Possible overfitting!
- Fix by using Laplace smoothing:
  - Adds 1 to each count

$$P(X_j = v \mid Y = k\ ) = \frac{\mathrm{c}_v + 1}{\displaystyle\sum_{v' \in \mathrm{values}(X_j)} \mathrm{c}_{v'} + |\mathrm{values}(X_j)|}$$

where

- c_v is the count of training instances with a value of v for attribute j and class label k
- $|\mathrm{values}(X_j)|$ is the number of values X_j can take on

Using the Naïve Bayes Classifier

- Now, we have

$$\mathrm{P}[Y = k | X = x] = \frac{\mathrm{P}[Y = k]\mathrm{P}[X_1 = x_1 \wedge \cdots \wedge X_d = x_d | Y = k]}{\boxed{\mathrm{P}[X_1 = x_1 \wedge \cdots \wedge X_d = x_d]}}$$

This is constant for a given instance, and so irrelevant to our prediction

– In practice, we use log-probabilities to prevent underflow

- To classify a new point $\mathbf{x}$,

$$h(\mathbf{x}) = \arg\max_{y_k} P(Y = k) \prod_{j=1}^{d} P(X_j = \underbrace{x_j}_{j^{\text{th}} \text{ attribute value of } \mathbf{x}} \mid Y = k)$$

$$= \arg\max_{y_k} \log P(Y = k) + \sum_{j=1}^{d} \log P(X_j = x_j \mid Y = k)$$

Naïve Bayes Classifier

- For each class label k
  1. Estimate prior $P[Y = k]$ from the data
  2. For each value v of attribute X_j
     - Estimate $\mathrm{P}[X_j = v | Y = k]$

- Classify a new point via:

$$h(\mathbf{x}) = \arg\max_{y_k} \log P(Y = k) + \sum_{j=1}^{d} \log P(X_j = x_j \mid Y = k)$$

- In practice, the independence assumption doesn't often hold true, but Naïve Bayes performs very well despite it

Computing Probabilities

- NB classifier gives predictions, not probabilities, because we ignore $\mathrm{P}(\mathrm{X})$ (the denominator in Bayes rule)
- Can produce probabilities by:
  - For each possible class label y_k, compute

$$\underbrace{\tilde{P}(Y = k \mid X = \mathbf{x})}_{\text{This is the numerator of Bayes rule, and is therefore off the true probability by a factor of } \alpha \text{ that makes probabilities sum to 1}} = P(Y = k) \prod_{j=1}^{d} P(X_j = x_j \mid Y = k)$$

  - α is given by $\alpha = \dfrac{1}{\sum_{k=1}^{\#classes} \tilde{P}(Y = k \mid X = \mathbf{x})}$
  - Class probability is given by

$$P(Y = k \mid X = \mathbf{x}) = \alpha \tilde{P}(Y = k \mid X = \mathbf{x})$$

Handling Continuous Features

- Use histograms
- Estimate $P\left[X_j = v \,\middle|\, Y = k\right]$ with normal distribution
  - Called Gaussian Naïve Bayes
- Use KDE
  - Uni-variate Kernel Density Estimate for $P\left[X_j = v \,\middle|\, Y = k\right]$
  - Multi-variate Kernel Density Estimate for $\mathrm{P}[X_1 = x_1 \wedge \cdots \wedge X_d = x_d | Y = k]$

Comparison to LDA

- Similarity to LDA
  - Both are generative models
  - They both estimate:

    $P[X = x \text{ and } Y = k] = P[X = x | Y = k] P[Y = k]$
- Difference from LDA
  - LDA uses multi-variate normal
  - LDA assumes same co-variances for all classes
  - Naïve Bayes make the conditional independence assumption

Naïve Bayes Summary

Advantages:

- Fast to train (single scan through data)
- Fast to classify
- Not sensitive to irrelevant features
- Handles real and discrete data
- Handles streaming data well

Disadvantages:

- Assumes independence of features

Document Classification

Test Data:

"planning
language
proof
intelligence"

PROBLEM SETTING

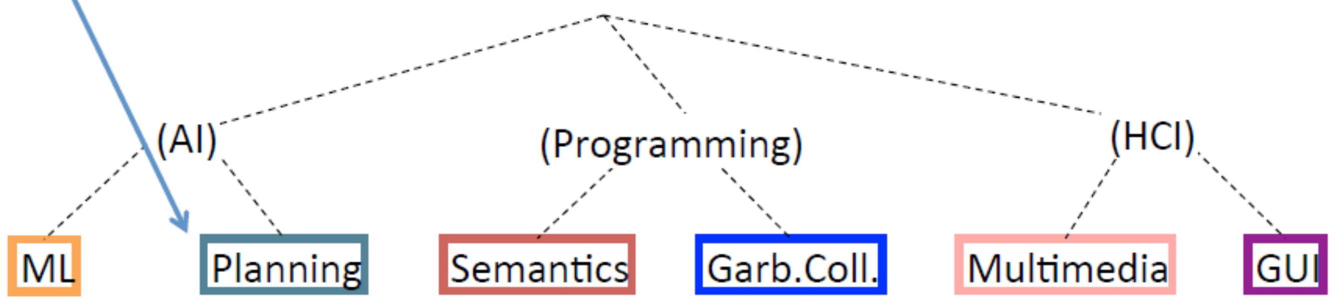
Given:

- Representation of a document
- Set of classes 1,…,K

Determine:

- Class to which document d belongs

Classes:



Training Data:

ML	Planning	Semantics	Garb.Coll.	Multimedia	GUI
learning	planning	programming	garbage	…	…
intelligence	temporal	semantics	collection		
algorithm	reasoning	language	memory		
reinforcement	plan	proof…	optimization		
network…	language…		region…		

Text Classification: Examples

- Classify news stories as *World, US, Business, SciTech, Sports, etc.*
- Add terms to Medline abstracts (e.g. "Conscious Sedation" [E03.250])
- Classify business names by industry
- Classify student essays as *A/B/C/D/F*
- Classify email as *Spam/Other*
- Classify email to tech staff as *Mac/Windows/ ...*
- Classify pdf files as *ResearchPaper/Other*
- Determine authorship of documents
- Classify movie reviews as *Favorable/Unfavorable/Neutral*
- Classify technical papers as *Interesting/Uninteresting*
- Classify jokes as *Funny/NotFunny*
- Classify websites of companies by Standard Industrial Classification (SIC) code

Bag of Words Representation

What is the ~~best~~ representation for documents?

simplest, yet useful

Idea: Treat each document as a sequence of words

- Assume that word positions are generated *independently*

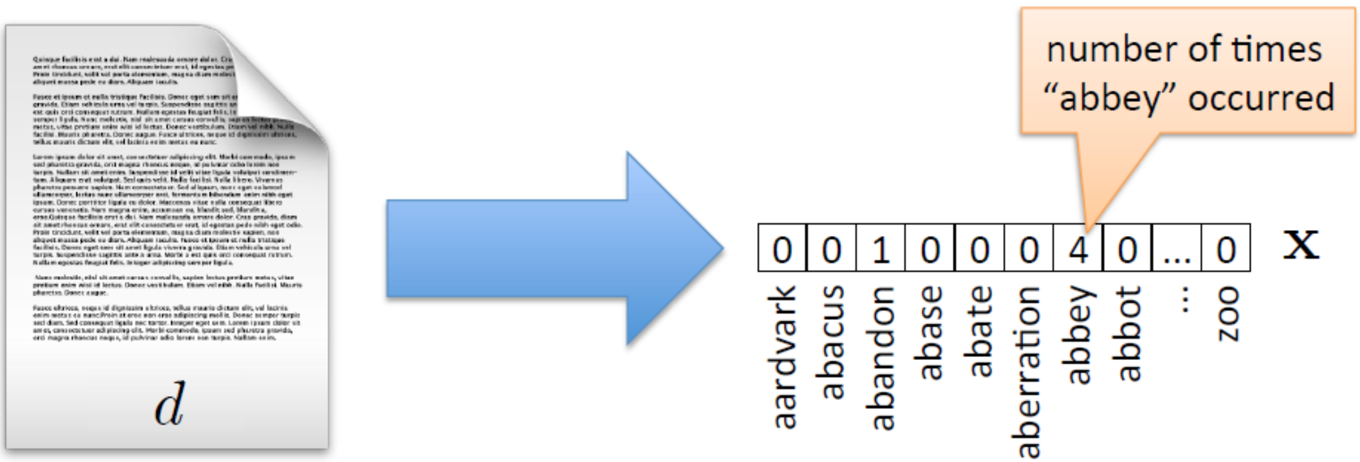
Dictionary: set of all possible words

- Compute over set of documents
- Use Webster's dictionary, etc.

Bag of Words Representation

Represent document d as a vector of word counts $\mathbf{x}$

- x_j represents the count of word j in the document
  - $\mathbf{x}$ is sparse (few non-zero entries)



Another View of Naïve Bayes

- Let the model parameters for class c be given by:

$$\boldsymbol{\theta}_c = \{\theta_{c1}, \theta_{c2}, \ldots, \theta_{c\underbrace{|D|}_{\text{size of dictionary } D}}\}$$

  - $\theta_{cj} = \mathrm{P}($word j occurs in a document from $c)$
  - Also have that $\sum_j \theta_{cj} = 1$

- The likelihood of a document d characterized by $\mathbf{x}$ is

$$P(d \mid \boldsymbol{\theta}_c) = \frac{(\sum_j x_j)!}{\prod_j x_j!} \prod_j (\theta_{cj})^{x_j}$$

  - This is just the multinomial distribution, a generalization of the binomial distribution $\binom{n}{k} p^k (1-p)^{n-k}$

Another View of Naïve Bayes

- The likelihood of a document d characterized by $\mathbf{x}$ is

$$P(d \mid \boldsymbol{\theta}_c) = \frac{(\sum_j x_j)!}{\prod_j x_j!} \prod_j (\theta_{cj})^{x_j}$$

- Use Bayes rule:

introduce class priors

$$\log P(\boldsymbol{\theta}_c \mid d) \propto \log \left(P(\boldsymbol{\theta}_c) \prod_{j=1}^{|D|} (\theta_{cj})^{x_j} \right) = \log P(\boldsymbol{\theta}_c) + \sum_{j=1}^{|D|} x_j \log \theta_{cj}$$

Therefore,

$$h(d) = \arg\max_c \left(\log P(\boldsymbol{\theta}_c) + \sum_{j=1}^{|D|} x_j \log \theta_{cj} \right)$$

This is just a linear decision function!

Document Classification with Naïve Bayes

1. Compute dictionary D over training set (if not given)
2. Represent training documents as bags of words over D
3. Estimate class priors via counting
4. Estimate conditional probabilities as $\hat{\theta}_{cj} = \dfrac{N_{cj} + 1}{N_c + |D|}$
   - N_{cj} is number of times word j occurs in documents from class c
   - N_c is total number of words in all documents from class c

- Naïve Bayes model for new documents (represented in D) is:

$$h(d) = \arg\max_c \left(\log P(c) + \sum_j x_j \hat{w}_{cj} \right)$$

$$\text{where } \hat{w}_{cj} = \log \hat{\theta}_{cj}$$

Review Naïve Bayes

- Density Estimators can estimate joint probability distribution from data
- Risk of overfitting and curse of dimensionality
- Naïve Bayes
  - Generative model (like LDA)
  - Reduces the complexity of density estimation
  - Assumes that features are independent given labels
- Applications: text classification with bag-of-words representation
  - Multinomial Naïve Bayes becomes a linear classifier

Acknowledgements

- Slides made using resources from:
  - Andrew Ng
  - Eric Eaton
  - David Sontag
  - Andrew Moore
- Thanks!